On the Classification of Orbifold del Pezzo Surfaces

by

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Declarations

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification.
I confirm that the work submitted is entirely my own work, except where work has formed part of jointly–authored contribuition has been included; the collaborative contributions have been indicated clearly and acknowledged. I confirm that appropriate credit has been given within this thesis where reference has been made to the work of others.

Introduction

A del Pezzo surface X is a smooth complex projective variety of dimension 2 whose anticanonical bundle is ample. From the associated graded ring, we can describe the variey X as the projective scheme

$$X = \mathrm{Proj}\big(\bigoplus_{m\geq 0} H^0(X, -mK_X)\big) \tag{1}$$

The classification of del Pezzo surfaces embodies a very classical topic. Indeed, it has been known since the 19th century that such surfaces appear as blow ups of $\mathbb{P}^2$ in $9-d$ ($d \leq 9$) points, where $d = (-K_X)^2$, or $\mathbb{P}^1 \times \mathbb{P}^1$.

When attempting to extend to the singular case we are still missing a classification just as elegant. Nevertheless, if we restrict to the case of *cyclic quotient singularities* a lot can be said about the structure of such varieties. These kinds of singularities are isolated points arising from an effective action of a finite cyclic group μ_N on the variety X such that analytically they are locally isomorphic to $\mathbb{C}^2/\mu_N$; varieties admitting this kind of singularities are called *orbifolds* and are log terminal and $\mathbb{Q}$-Gorenstein ([KSB88], [Kaw88]). We can subdivide cyclic quotient singularities in two classes, namely *T-singularities* and *R-singularities.* The former are characterised by being smoothable for a specific kind of deformation ($\mathbb{Q}$–*Gorenstein*, or *qG-deformation*), while the latter are rigid. This type of deformation is unobstructed for cyclic quotient singularities and preserves a number of geometric and topological invariants, providing a great deal of information about the associated graded ring.

In this work, we will focus on surfaces admitting two specific kinds of such singularities, namely $\frac{1}{3}(1,1)$ and $\frac{1}{5}(1,2)$, which are qG–rigid. The interest in classifying surfaces admitting these specific types of singularities comes from two previous works: the construction of cascades of unprojections by Reid and Suzuki ([RS03]) and the classification via toric degenerations of del Pezzo surfaces with $\frac{1}{3}(1,1)$ singularities by Corti and Heuberger ([CH15]).

In the former, the authors use graded ring methods to find explicit birational constructions of surfaces with the aforementioned singularity type, obtaining a singular analogoue to the classical construction by del Pezzo. Indeed, their construction consists of obtaining varieties from a base one by connecting them with *unprojections*. The theory behind unprojections has been developed initially by Kustin and Miller ([KM83]) in the 1980s and refined by Reid, Brown and Papadakis ([Reid00], [RP04], [BKR], [Pap01]) in an attempt to find structure theorems for Gorenstein rings in codimension 4. Roughly speaking, the crucial aspect of these constructions is the fact that they allow us to create new varities from old ones at the price of increasing the codimension of the embedded variety.

On the other hand, in [CH15] the authors find 29 qG–deformation families of surfaces with rigid content $\frac{1}{3}(1,1)$, 26 of whom admit toric degenerations. Moreover, they give a biregular classification of all the families involved consisting of complete intersections in toric varieties, (weighted) Grassmannians and more complicated examples, such as (weighted) degeneracy loci. The key link with the work previously mentioned is the subdivision of these families in cascades of unprojections in the sense of [RS03].
Thus the interplay between these two aspects represents the main motivation behind this work: our main aim is to find a complete classification of orbifold del Pezzo surfaces up to qG–deformation with $\frac{1}{3}(1,1)$ and $\frac{1}{5}(1,2)$ points by putting together graded ring methods and birational constructions with toric degenerations.

Our approach will consist of several steps: firstly we will look at the graded ring of the said del Pezzo surfaces in order to analyse the numerical invariants. Indeed, as we are dealing with $\mathbb{Q}$–Gorenstein rings, the singularity type allows us to easily compute such invariants ([Reid85], [Reid1] [BRZ13]) and to find a bound for the number of singularities.

Secondly, we wish to find birational models for our surfaces; to this end we will study the structure of these varieties in terms of *Mori Theory*. Indeed, our aim is to construct a Minimal Model Program (MMP in short) for our singular surfaces by means of *extremal contractions*. This will allow us to generalise the classical approach of the Italian school of Castelnuovo et al.: it is well known that for smooth complex projective surfaces the MMP consists of contractions of rational curves with self intersection (-1). After a finite number of contractions we end up with either a surface with nef canonical class or a Mori Fibre Space. In the singular case, we can still associate to our initial surface a model surface by means of birational maps: these can be factored into extremal contractions that depend on the curve configuration of the minimal resolution of our initial surface. These contractions involve a specific class of curves, namely *extremal rays* of the Mori Cone. The key feature in the proof of existence of such del Pezzo orbifolds is describing a suitable Directed Minimal Model Program by analysing these extremal rays (and hence curve configurations of the minimal resolution of such surfaces). Ultimately, via a case–by–case analysis, we can determine the possible surfaces that can arise from these birational contructions. Thus, we obtain our first main result:

Theorem 0.0.1 (Main Theorem 1)**.** *There are 33 isomorphism classes of minimal del Pezzo Orbifolds with $h^0(-K_X) \neq 0$ admitting $\{k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2)\}$ singularities (where $k_2 \geq 1$).*
The curve configurations of their resolutions are listed in Table 2.5.
Their birational models are recapped in Table 5.1.

Subsequently, we want to relate these *minimal surfaces* to their qG–deformation classes. In this instance, the toric degenerations come into play. When considering toric surfaces, the notion of qG-deformation is strictly related to the Fano polygon associated to the variety, and involves *mutations*: two polygons that are mutation equivalent define two surfaces belonging to the same qG–deformation class ([AK15], [Il12], [ACHK15]). Via computer algebra, it is possible to check whether two polygons are mutation equivalent, thus given a specific singularity content we can have a complete list of toric representatives for qG–classes ([KNP15], [CK17]).

In order to find a connection between the minimal surfaces and the (possible) toric degenerations we turn to the deformation theory of *T–varieties*: we know ([Il12], [ACHK15]) that the general element in the special pencil representing the qG–deformation defined by a mutation is a (not necessarily toric) variety that inherits a $(\mathbb{C}^\times)$ action from the special fibres. These types of varieties present a very combinatorial nature; therefore, we can exploit this property to determine their qG–deformations ([Il09], [IV11]). More precisely, the local toric nature gives an explicit deformation of a cyclic quotient singularity (when possible); consequently, the Gross–Siebert program assures us that we can glue these local deformations together ([GS11], [Pri18]). As a result, we can link our toric candidates to the minimal surfaces found from the MMP, which would represent their smoothings.

Finally, we are able to link all of the qG–deformation classes admitting toric degenerations to the cascade constructions. Via these methods we can eventually give an exhaustive classification. Ultimately, we have construction theorems summing up the cascade structure for every singularity type. For instance, we have the following:

Theorem 0.0.2 (qG–deformation classes for surfaces with $\frac{1}{5}(1,2)$ point)**.** *There are 9 qG-deformation classes of del Pezzo surfaces with $1 \times \frac{1}{5}(1,2)$ orbifold point:*

- *if $K_S^2 = \frac{32}{5}$, then $S = T_6 \subset \mathbb{P}(1,1,3,5)$ and it is minimal;*
- *if $K_S^2 = \frac{27}{5}$, then either $S = T^{(1)} \subset \mathbb{P}(1^6, 2^2, 3, 4, 5)$ or $S = V_6 \subset \mathbb{P}(1,1,2,5)$, where V_6 is minimal;*
- *if $K_S^2 = \frac{22}{5}$, then either $S = T^{(2)} \subset \mathbb{P}(1^5, 2^2, 3, 4, 5)$ or $S = V^{(1)}$;*
- *if $\frac{2}{5} < K_S^2 \leq \frac{17}{5}$, then $d = 3,4,5$ and $S = T^{(d)} = V^{(d-1)} \subset \mathbb{P}(1^{7-d}, 2^2, 3, 4, 5)$;*
- *if $K_S^2 = \frac{2}{5}$, then $S = T^{(6)} = V^{(5)} = T_{6,8} \subset \mathbb{P}(1,2,3,4,5)$;*

where $T^{(d)} = Bl_{P_1,\ldots,P_d}(T_6)$ and $V^{(d-1)} = Bl_{Q_1,\ldots,Q_{d-1}}(V_6)$ represent blow–ups of the minimal surface at respectively d or $d-1$ general points.

Not only does this construction shows a neat description of all the qG–families for a given singularity type, but it gives us a precise count of all the qG–classes that can admit a toric degeneration as well. Ultimately, we have:

Theorem 0.0.3 (Main Theorem 2)**.** *Let S be a del Pezzo surface with singularity content*

$$(n, k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2)).$$

where $k_2 \geq 1$. Then there are 69 qG–classes of such surfaces admitting a toric degeneration

It turns out that not only are toric degenerations useful to classify such del Pezzo surfaces, but they also play a crucial role in Mirror Symmetry. In the case of surfaces, for instance, it has been conjectured ([ACHK15]) that there is a *mirror map* linking the quantum period of the surface X with the classical period of the Fano polygon associated to the toric degeneration of X. This depends on the existence of *maximally mutable Laurent polynomials*([KNP15]), a notion strictly intertwined with mutation classes of Fano polygons. The authors have developed a bunch of conjectures that play around with these elements with the aim to classify a wide class of del Pezzo Orbifolds. The main goal (conjectured again in [ACHK15]) is the following: if X is a del Pezzo orbifold admitting a toric degeneration (i.e. of *class TG*), then there is a one to one correspondence between

$$\begin{pmatrix} \text{Fano polygons} \\ \text{up to mutation} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{qG-deformation classes of locally qG-rigid} \\ \text{del Pezzo surfaces of class TG} \\ \text{w/ cyclic quotient singularities} \end{pmatrix}$$

Evidence for this conjecture has been given for the smooth case ([KNP15], [HP10]) and for rigid singularities of type $\frac{1}{3}(1,1)$ ([CH15]). From our account we obtain another confirmation for this conjecture, but our results raise a further point: which del Pezzo surfaces are of class TG?
Moreover, our results give a description of the qG–deformation families in cases for which the surface does admit a toric boundary (i.e. $h^0(-K_X) \neq 0$). Thus, it would be interesting to see if the methods we used to construct deformations of T–varieties can be applied to a wider class of surfaces.

Plan of the work

Chapter 1 is devoted to outlining the problem. We introduce the background material, namely define orbifold del Pezzo surfaces, qG–deformations and graded ring methods. Following this, we use the properties of these classes of surfaces to compute numerical invariants and find a bound for the number of singularities. Ultimately, we obtain a list of numerical candidates for our surfaces.

In Chapter 2 we recall some aspects of the Mori theory for surfaces, we define the notion of minimal surfaces and we find surfaces with singularity content $(n, k_1 \times \frac{1}{5}(1,2) + k_2 \times \frac{1}{3}(1,1))$ having Picard rank $\rho = 1$. Later we establish a Directed Minimal Model Program for our class of surfaces and by analysing our numerical candidates we find the isomorphism classes of our del Pezzo surfaces.

In Chapter 3 we discuss the toric case: we find all of the possible mutation classes of our orbifolds and we introduce the formalism of T–varieties. We then show how to link qG–deformations to equivariant complexity 1 deformations. We give a couple of enlightening examples to better understand the complexity 1 enviroment and deformations.

In Chapter 4 we finally construct the cascades from the representatives of the qG–classes and we give a complete count of all the deformation classes for our type of surfaces.

Chapter 5 contains tables representing a summary of the MMP outcomes and the classification of toric surfaces representing the mutation classes.

Lastly, in the Appendix we report the calculations that lead us to the classification of the isomorphism classes in Chapter 2.

Notation

All the varieties are assumed to be complex and projective unless otherwise stated.

X, S - Orbifold del Pezzo surface
$\mathcal{O}_X$ - Structure sheaf of X
K_X - Canonical class of X
$\omega_X = \mathcal{O}_X(K_X)$ - canonical sheaf of X
K_X^2 - degree of X
$\frac{1}{N}(a,b)$, $\frac{1}{N}(1,a)$ - cyclic quotient singularity
$\varphi : Y \to X$ - minimal resolution of X
$\mathrm{Sing}(X)$ - singular locus of X
$\mathcal{B}$ - basket of singularities
$n = e(X^0)$ - topological Euler number of the smooth locus X^0
$h^0(X,D) = h^0(D) = \dim H^0(X, \mathcal{O}_X(D))$ - dimension of Riemann–Roch space
for a divisor $D \subset X$
$\mathrm{Pic}(X)$ - Picard group of X
$\rho(X) = \dim Pic(X)$ - Picard rank of X
$\mathrm{NS}(X)$ - Néron–Severi group of X
$\mathrm{NE}(X)$ - cone of effective divisors of X
$\psi : X \to X'$ - extremal contraction
$k(X)$ - Kodaira dimension of X
$[b_1, \dots, b_l]$ - Hirzebruch–Jung continued fraction
(k_1, k_2) - type of R–content of a orbifold del Pezzo surface with basket of singularities
$\mathcal{B} = \{k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2)\}$
$(n, \mathcal{B})$ - singularity content
$S^{n,i}_{(k_1,k_2)}$ - i–th surface with singularity content $(n, k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2))$
$X^{n,i}_{(k_1,k_2)}$ - i–th toric surface with singularity content
$(n, k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2))$
$\pi : \mathcal{X} \to B$ - flat family of complex projective surfaces over a base space B
$\mathrm{Def}_{qG}(X)$ - component of qG–deformations in the versal space $\mathrm{Def}(X)$
$N', M' \cong \mathbb{Z}^2$, $N, M \cong \mathbb{Z}$ - lattices
P - Fano polygon
σ - cone in $N' \otimes \mathbb{Q}$
Σ - Fan in $N' \otimes \mathbb{Q}$
Δ - polyhedron
$\mathrm{tail}(\Delta)$ - tailcone of Δ
$\mathcal{D}$ - polyhedral divisor

$\mathrm{Pol}^+_\sigma(N)$ - semigroup of polyhedral divisors w/ tailcone $\mathrm{tail}(\Delta) = \sigma$ in the lattice N with respect to the Minkowski sum

$X(\mathcal{D})$ - affine scheme obtained from the polyhedral divisor $\mathcal{D}$

$\mathcal{S}$ - divisorial fan

$X(\mathcal{S})$ -complete variety obtained from the divisorial fan

$K_l(X_{(n,q)})$ - set of P–resolutions of the quotient singularity $\frac{1}{n}(1, q)$

$[m = \lambda]$ - level set with $v \in N'$ such that $\langle v, m\rangle = \lambda$

Chapter 1

Setting

In this Chapter, we describe some of the background material necessary to approach the problem regarding the graded ring of del Pezzo surfaces admitting orbifold points. We introduce the main tools of our constructions, namely unprojections and qG–deformations. Ultimately, we analyse their invariants finding bounds for the number of singularities such surfaces can admit, and we list the possible cases.

1.1 Orbifold Del Pezzo Surfaces

Let X be a complex projective variety of dimension 2.

Definition 1.1.1. The surface X is said to have a **cyclic quotient singularity** if it admits an isolated point $p \in X$ such that a neighbourhood of the point in the classical topology is analytically isomorphic to $\mathbb{C}^2/\boldsymbol{\mu}_N$ where $\boldsymbol{\mu}_N$ is a cyclic group of order N and its action on such open affine neighbourhood is defined by:

$$\mu_N \ : (x, y) \longmapsto (\zeta^a x, \zeta^b y)$$

where ζ is a primitive N–th root of unity.
We denote the cyclic quotient singularity by $\frac{1}{N}(a, b)$ (with $(a, N) = (b, N) = 1$); in particular, the action can be rescaled so that every cyclic quotient singularity corresponds to a $\frac{1}{N}(1, a)$ point.

Definition 1.1.2 ([AK15], Definition 2.5)**.** The singularity $p = \frac{1}{N}(1, a)$ is said to be:

- A **T-singularity** if it is of the form $\frac{1}{nr^2}(1, nra - 1)$;
- A **R-singularity** if it is of the form $\frac{1}{\omega_0 r}(1, \omega_0 a - 1)$.

for some positive integers r, ω_0. More precisely, a cyclic quotient singularity of the surface X can be written as $\frac{1}{\omega r}(1, \omega a - 1)$, where $\omega = nr + \omega_0$ with $0 \leq \omega_0 < r$; in this case, we call $\frac{1}{\omega_0 r}(1, \omega_0 a - 1)$ the **R-content** (or **residual content**) of the singular surface X, and it is denoted by $\mathrm{res}(p)$.

Definition 1.1.3. An **Orbifold del Pezzo surface** X is a complex projective surface with ample anticanonical class $-K_X$ and a finite number of cyclic quotient singularities. Moreover, X admits a finite number of (-1)-curves. In particular, K_X is a $\mathbb{Q}$-Cartier divisor, so there exists a rational number f (the so–called Fano index) such that if A is a primitive ample class, then $-K_X = fA$.

Generically speaking, if X is a normal variety and $\varphi : Y \to X$ its minimal resolution, then

$$K_Y = \varphi^*(K_X) + \sum a_i E_i \tag{1.1}$$

where K_X, K_Y denote the canonical divisors of the surfaces, and E_i the exceptional divisors appearing from the resolution.

Definition 1.1.4. The variety X as above is said to have:

- **log canonical singularities** if $a_i \geq -1$;
- **log terminal singularities** if $a_i > -1$;
- **canonical singularities** if $a_i \geq 0$;
- **terminal singularities** if $a_i > 0$

In particular, a normal surface singularity $p \in X$ is log terminal if and only if it is a quotient singularity ([Kaw88]). Such surfaces are often referred as *log del Pezzo surfaces* (or LDP) and have been classified for index ≤ 3 by various authors (see, for instance, [Nik90], [Nik89], [Nik90], [AN06] and [FY17]).

In the case of *toric surfaces*, i.e. Zariski closures of torus embeddings ([CLS11],[Ful93]), a complete classification of isomorphism classes up to index 17 appears in the online Graded Ring DataBase (GRDB), see [BK] and [KKN08] for more deatils.
Indeed, given the combinatorial nature of such varieties, we have a correspondence between toric LDPs and a specific class of polygons.

Definition 1.1.5. Let $N' \cong \mathbb{Z}^2$ be a lattice, and let $P \subset N' \otimes \mathbb{Q} = N'_{\mathbb{Q}}$ be a lattice polygon. Then P is said to be a **LDP–polygon** if the origin $0 \in N'$ is a strict interior point and if all of its vertices are primitive.

Moreover, there is a one–to–one correspondence between LDP-polygons as defined above and toric log del Pezzo surfaces. Thus, every toric LDP arises this way.

Lastly, we recall another property of such surfaces which will come in use when dealing with their invariants.

Definition 1.1.6. A variety X is said to be $\mathbb{Q}$–**Gorenstein** if it is Cohen–Macaulay and K_X is $\mathbb{Q}$–Cartier, i.e. there exists an $m \in \mathbb{Z}$ such that mK_X is Cartier.

For a normal projective $\mathbb{Q}$–Gorenstein variety, the graded ring associated to an ample $\mathbb{Q}$-Cartier divisor D

$$R(X, D) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD))$$

is finitely generated and gives an embedding

$$X \cong \operatorname{Proj}(R(X, D)) \hookrightarrow \mathbb{P}(m_0, \ldots, m_N) \tag{1.2}$$

into a weighted projective space $\mathbb{P}(m_0, \ldots, m_N)$ for some $N \in \mathbb{N}$ and weights $m_i \in \mathbb{N}$ ([Reid85]).

As we will see, this structure has interesting consequences on the invariants: if X is a del Pezzo orbifold with some isolated cyclic quotient singularities such that, if the singular locus is represented by the set

$$\operatorname{Sing}(X) = \{p_i = \tfrac{1}{N_i}(1, a_i)\}$$

then the polarisation induced by $-K_X$ makes it a $\mathbb{Q}$–Gorenstein surface and its Hilbert series

$$P_X(t) = \sum_{m \geq 0} h^0(mD) t^m \tag{1.3}$$

can be calculated in terms of the so–called *Ice Cream Functions* ([BRZ13]): as in 1.2, let i denote the embedding of X into the weighted projective space $\mathbb{P} = \mathbb{P}(m_0, \ldots, m_N)$;

then, by Hilbert syzygy theorem ([Eis11]), there exists a finite free resolution

$$0 \longleftarrow i_*\mathcal{O}_X \longleftarrow M_0 \longleftarrow M_1 \longleftarrow \cdots \longleftarrow M_c \longleftarrow 0$$

where $M_i \cong \bigoplus_j \mathcal{O}_{\mathbb{P}}(b_{i,j})$ are free graded modules over $\mathbb{C}[x_0, \dots, x_N]$, $M_0 = \mathcal{O}_{\mathbb{P}}$ and every morphism $M_{i+1} \to M_i$ is represented by a matrix with homogeneous entries. As X is $\mathbb{Q}$–Gorenstein, the lenght c of the sequence is equal to the codimension of X in $\mathbb{P}$ and $M_c \cong \mathcal{O}_{\mathbb{P}}(-k_X - \sum m_i)$ (the m_i's represent the weights of the coordinates x_i's of the weighted projective space $\mathbb{P}$), and the number k_X is the *canonical weight* of X.
Then 1.3 can be written as

$$P_X(t) = P_I(t) + \sum_{p \in \mathrm{Sing(X)}} P_{\mathrm{orb}}(p, k_X)(t) \tag{1.4}$$

where P_I is the *Initial* part (which corresponds to $P_X(t)$ until degree $\lfloor \frac{c}{2} \rfloor$) and P_{orb}, i.e. the Ice cream functions, correspond to the contributions of singularities and they depend on the singularity type of p (for more detail see [BRZ13]).

Moreover, the contributions of the orbifold singularities are traced by the *Orbifold Riemann–Roch formula*, introduced by Reid in [Reid85]:

$$\chi(X, D) = \mathrm{RR}(X, D) + \sum_{p \in \mathrm{Sing}(X)} c_p(D) \tag{1.5}$$

where $\mathrm{RR}(X, D)$ denotes the contribuition of the classical Riemann–Roch, and $c_p(D)$ denote the fractional contributions of the cyclic quotient singularities, and they are calculated in terms of Dedekind sums (for details, see [Reid85]).
The two formulas are strictly intertwined, and we can recover 1.5 by manipulating the Hilbert series 1.4. We will discuss this further in Section 1.5.

From now on every surface is supposed to be an Orbifold del Pezzo surface unless otherwise stated.

1.2 qG-deformations

The notion of *qG–deformation* of cyclic quotient singularities is discussed in detail in [KSB88] and concerns flat families of normal surfaces. Specifically, consider X_0 a surface quotient singularity, and $\mathcal{X} \to D$ a a one–parameter deformation of X_0.

Definition 1.2.1. The flat family $\mathcal{X} \to D$ is said to be $\mathbb{Q}$–Gorenstein if the total space $\mathcal{X}$ is $\mathbb{Q}$–Gorenstein.

Consider a finite base change

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X} \times_D D' \arrow[l] \arrow[d] \\
\{pt\} \arrow[r, hook] & D & D' \arrow[l]
\end{tikzcd}$$

Then from [KSB88] we know that $\mathcal{X}' = \mathcal{X} \times_D D'$ is $\mathbb{Q}$-Gorenstein if and only if $\mathcal{X}'$ is. As $\mathcal{X}$ is $\mathbb{Q}$–Gorenstein, the *covering trick* described by Reid in [Reid80] assures us that there exists a cyclic cover of the total space $\mathcal{Z} \to \mathcal{X}$ that is canonical. Thus, by Brieskorn-Tyurina theory of simultaneous resolutions, we have that there exists a simultaneous resolution of the family $\mathcal{Y} \to D$ so that $g : \mathcal{Y} \to \mathcal{Z}$ is a morphism and, at every fibre over D, $g_t : Y_t \to Z_t$ is the minimal resolution of the Du Val singularities of Z_t.
Thus, we have:

Theorem 1.2.1 ([KSB88], Theorem 3.5)**.** *If $\mathcal{X} \to D$ is a one–paramenter deformation of (X_0, p), then, up to base change, there exists a birational morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ such that over a general point $\varphi : Y_t \to X_t$ is the minimal resolution, and the special fibre Y_0 is normal with quotient singularities.*
In particular, $K^2_{X_t}$ is locally constant on the base.

We have seen in Chapter 1 that we can subdivide cyclic quotient singularities in two main families, namely R and T–singularities. The property of T–singularities is that they admit a one–parameter qG–smoothing: in that case, the general element X_t is smooth and the total space $\mathcal{X}$ admits one terminal singularity at the origin. In particular, in terms of residual content we have:

Propostion 1.2.1 ([AK15], Proposition 2.8)**.** *If $p \in X$ is a cyclic quotient singularity, then it admits a qG–smoothing if and only if $res(p) = 0$.*
If $res(p) \neq \emptyset$, then there exists a qG–deformation of X such that the general fibre of the family has a cyclic quotient singularity with residual content $res(p)$.

This is particularly useful to understand the versal deformation space $\mathrm{Def}(X_0)$: if (X_0, p) denotes the germ of a T–singularity, then the component of qG–deformations $\mathrm{Def}_{qG}(X_0) \subset \mathrm{Def}(X_0)$ is an irreducible component of the versal deformations space.
Moreover, in [KSB88] the authors introduce the notion of P–resolution, i.e. a partial

resolution of the quotient singularity $\psi : Z_0 \to X_0$ where K_{Z_0} is ample relative to ψ, and Z_0 admits singularities of class T only. One of the main results of their paper is the link between these resolutions and the components of $\mathrm{Def}(X_0)$:

Theorem 1.2.2. *There exists a one–to–one correspondence between P–resolutions of the quotient singularity X_0 and the components of* $\mathrm{Def}(X_0)$.

Altman ([Alt98]) and Ilten ([Il09]) have treated in lenght the case of this correspondence in the case of toric surfaces; moreover, their techniques can be used to establish qG–deformations of surfaces with cyclic quotient singularities, and we will discuss this in detail in Chapter 4.

Due to the toric nature of the local behaviour of such singularities, it is possible to recover the invariants of our surfaces, which depend on the singularity type (and specifically, on the residual content).
Indeed, if $p \in X$ is a cyclic quotient singularity of type $\frac{1}{r}(1, a-1)$ on a log del Pezzo X, then the Hirzebruch–Jung continued fraction ([Reid1])

$$\frac{r}{(a-1)} = [b_1, \ldots, b_l] \tag{1.6}$$

gives information on the minimal resolution of X. Let $\varphi : Y \to X$ be the minimal resolution of the surface X and $E_i \subset Y$ denote the exceptional curves of such resolution. Then, over the singular point p, the exceptional curves form a chain where each curve E_i has self intersection $E_i^2 = -b_i$ for every $i = 1..l$ and intersects another curve E_j (transversely) only if $j = i - 1$ or $i + 1$.
Moreover, if K_X, K_Y denote the canonical classes of the surfaces X and Y respectively, then

$$K_Y = \varphi^*(K_X) + \sum_i d_i E_i \tag{1.7}$$

where the discrepancies $d_i \in \mathbb{Q}$ are functions of the numbers b_i.
As a result, we obtain a formula for the degree of the log del Pezzo X in relation to the singularity type of the point p. In particular, for T–singularities the discrepancy is void and for singularities having residual content $\mathrm{res}(p) \neq \emptyset$ then the discrepancies are calculated only in function of $\mathrm{res}(p)$. Thus, we can introduce the following:

Definition 1.2.2 ([AK15], Definition 3.1)**.** The **Singularity Content** of the surface X is the pair $(n, \mathcal{B})$, where $n = e(X^0)$ is the topological Euler number of X^0, the non-singular locus of X, and $\mathcal{B}$ is the basket of residual singularities of X.

Theorem 1.2.3 (Noether formula for orbifolds, [AK15], Proposition 3.3). *If X is a orbifold del Pezzo surface with singularity content $(n, \mathcal{B})$, then*

$$K_X^2 = 12 - n - \sum_{p \in \mathcal{B}} A_p \tag{1.8}$$

where $n = e(X^0)$ and

$$A_p = l_p + 1 - \sum_{i=1}^{l_p} d_i^2 b_i + 2 \sum_{i=1}^{l_p - 1} d_i d_{i+1}$$

with l_p denoting the lenght of the Hirzebruch–Jung continued fraction of the singularity $p \in X$.

As X admits cyclic quotient singularities only, then the topological Euler number can be calculated directly from the singularity type of the points in $\mathrm{Sing}(X)$: as we have seen in 1.1.2, any cyclic quotient singularity p_j can be factorised as $\frac{1}{\omega_i r_i}(1, \omega_i a_i - 1)$ where $\omega_i = n_i r_i + \omega_{0,i}$. Geometrically, this corresponds to a subdivision of the cone in a lattice $N' \cong \mathbb{Z}^2$ corresponding to the quotient singularity p_i in $n_i + 2$ subcones; in particular, every subcone will either correspond to a T-singularity or to a residual singularity. Ultimately, the topological Euler number corresponds to $n = \sum_i n_i$ and it is invariant under qG–deformation (see [AK15] for details).

Theorem 1.2.4 ([ACHK15], Theorem 6). *If X is a del Pezzo surface with $\{p_i \in X\}$ quotient singularities, then there exists a smooth morphism of deformation functors:*

$$\mathrm{Def}_{qG} X \to \prod \mathrm{Def}_{qG}(p_i, X) \tag{1.9}$$

where the former is the global and the latter is the local deformation functor.

Consequently, cyclic quotient singularities are unobstructed. In the case of del Pezzo orbifolds, this implies that if X_1,X_2 are two such surfaces that are qG–equivalent then there exists a flat family $\pi : \mathcal{X} \to B$ such that for $\lambda_1, \lambda_2 \in B$ then $\pi^{-1}(\lambda_1) = X_1$ and $\pi^{-1}(\lambda_2) = X_2$.
In the case of toric del Pezzo orbifold, the notion of *mutation* (see [KNP15] and [AK15]) gives an explicit description of qG–deformation of toric surfaces in terms of their Fano polygons (we will see this in more detail in Section 3.3).

Theorem 1.2.5. *[[Il12], Theorem 1.3 /[ACHK15], Theorem 7] Let P, P' two Fano polygons associated to the toric varieties $X_P, X_{P'}$, and suppose that there exists a mutation between the two polygons. Then there exists a qG-pencil $g : \mathcal{X} \to \mathbb{P}^1$ with scheme-theoretic fibres $g^*(0) = X_P$ and $g^*(\infty) = X'_P$.*

Moreover, the general fibre of such deformation is a variety that inherits a $(\mathbb{C}^\times)$ action but in general it is not toric. Such varieties are called *T–varieties* and will be discussed in detail in section 4.3.

1.3 Unprojections and Cascades

One of the most interesting constructions arising from this classification is the subdivision of qG–deformation classes into cascades. In this section we will thus introduce the main ingredient of this construction, i.e. unprojections. The theory of unprojections has been treated extensively in works of Reid (see for instance [Reid00]) and Papadakis ([RP04]), and it is particularly useful to find explicit contstructions of algebraic varieties. Check [BKR] or [Pap01] for treatments on structure theorems.

Roughly speaking, an unprojection is a birational map that serves as the inverse of a blow up. One of the easiest examples is given by the $Ax - By$ argument [Reid00]: let $X_3 \in \mathbb{P}^3$ be a cubic surface, and assume that X contains a line defined by the equations $x = y = 0$. As X is defined by one polynomial, say f of degree 3, then f must be of the form $Ax - By$, where A, B are polynomials of degree $degA = degB = 2$. Then we can define the new variable

$$s := \frac{A}{y} = \frac{B}{x} \tag{1.10}$$

where $wgt(s) = 1$. The ratios in 1.10 can be interpreted as the system of equations

$$\begin{cases} sy - A = 0 \\ sx - B = 0 \end{cases} \tag{1.11}$$

which gives the equations for a complete intersection $\tilde{X} \in \mathbb{P}^4$ with coordinates (x, y, z, s).Thus, adding the new variable s defines a birational map

$$\begin{gathered} X \dashrightarrow \tilde{X} \\ (x, y, z, w) \mapsto (x, y, z, w, s = \tfrac{A}{y} = \tfrac{B}{x}) \end{gathered} \tag{1.12}$$

having for birational inverse the natural projection from the point $p_s = (0,0,0,1)$. Moreover, the equations describe the contraction of the line D to the point p_s.

More generally, let X be a surface that is $\mathbb{Q}$-Gorenstein and let $D \subset X$ be a divisor that is also Q-Gorenstein. Then by Serre-Grothendieck duality we have

$$\omega_D = \mathcal{E}xt^1(\mathcal{O}_D, \omega_X) = H^1(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_D, \omega_X)) \tag{1.13}$$

So consider the short exact sequence:

$$0 \leftarrow \mathcal{O}_D \leftarrow \mathcal{O}_X \leftarrow \mathcal{I}_D \leftarrow 0 \tag{1.14}$$

Then, by applying the functor $\mathcal{H}om(-, \omega_X)$, we obtain

$$0 \to \omega_X \to \mathcal{H}om(\mathcal{I}_D, \omega_X) \to \omega_D \to 0 \tag{1.15}$$

where the last map represents the Poincaré residue map. As a result, $\mathcal{H}om(\mathcal{I}_D, \omega_X)$ as a $\mathcal{O}_X$-module is generated by two elements: i as a basis for ω_X, and s for $\mathcal{H}om(\mathcal{I}_D, \omega_X)$. In particular, we have that

1. $s : \mathcal{I}_D \to \omega_X$ is injective;
2. the image of s in ω_D is a basis of ω_D.

So $s : \mathcal{I} \to \mathcal{J} \subset \omega_X$ defines an isomorphism of ideal sheaves. If $I = (f_1, \dots, f_k)$ locally, then $s(f_i) := g_i$. Furthermore, we can think about s as a rational function so that $s = \dfrac{g_i}{f_i}$, and the map corresponds to the multiplication by the element s. In particular, s has poles on D.

Theorem 1.3.1 (Kustin-Miller). *Let s be the unprojection variable in the unprojection ring*

$$\mathcal{O}_X[s]/(sf_i - g_i) \tag{1.16}$$

then the ring is Gorenstein and s is a regular element in it.

The multiplication by the regular element s gives the birational map:

$$X \dashrightarrow \widetilde{X}$$

where $\widetilde{X}$ has codim$(\widetilde{X})$ =codim(X)+1, and its birational inverse is the natural projection of s. Geometrically, if $X \subset \mathbb{P}(a_1, \dots, a_N)$, then the map describes the contraction of the divisor D to a point $p_s = (0, \dots, 0, 1) \in \mathbb{P}(a_1, \dots, a_N, \mathrm{wgt}(s))$.

Definition 1.3.1. A ***cascade of unprojections*** is a chain of unprojections

$$\begin{tikzcd} X_0 \\ X_1 = \mathrm{Bl}(X_0) \arrow[u, dashed] \\ X_2 = \mathrm{Bl}(X_1) \arrow[u, dashed] \\ \vdots \arrow[u, dashed] \\ X_k = \mathrm{Bl}(X_{k-1}) \arrow[u, dashed] \end{tikzcd}$$

starting from a base variety X_0 and by subsequently blowing up X_0 at general points. Every map $X_i \dashrightarrow X_{i-1}$ is an unprojection where $\rho(X_i) = \rho(X_{i-1}) + 1$ and $\mathrm{codim}(X_i) = \mathrm{codim}(X_{i-1}) - 1$. These maps will give a sequence of birational maps which, from the variety X_k, contract rational (-1)-curves and give surfaces with increasing codimension.

The notation was introduced by Reid and Suzuki in [RS03]: the authors explicitly contruct cascades over $\mathbb{P}(1,1,3)$ and over the sextic $T_6 \subset \mathbb{P}(1,1,3,5)$. These surfaces are orbifold del Pezzo surfaces with respectively $\frac{1}{3}(1,1)$ and $\frac{1}{5}(1,2)$ singular points. Indeed, the case of $\mathbb{P}(1,1,3)$ fits into the classification carried out in [CH15] and mentioned in Section 1.4 below.
The cascade constructed for $T_6 \subset \mathbb{P}(1,1,3,5)$ will be discussed in Chapter 4 together with the full classification.

1.4 The case $k \times \frac{1}{3}(1,1)$

Surfaces having singulariy content $(n, \{k \times \frac{1}{3}(1,1)\})$ have been completely classified in [CH15] using toric degenerations. In particular, they are grouped in cascades as defined in [RS03].

Theorem 1.4.1 ([CH15], Theorem 12). *There are 29 qG-deformation families of del Pezzo surfaces with $k \geq 1$ $\frac{1}{3}(1,1)$ points. Exactly 26 of them admit a qG-degeneration to a toric surface.*

In particular the number of $\frac{1}{3}(1,1)$ singularities is at most 6, giving exactly 6 Cascades in the sense of Reid (cfr. [RS03]): The main results state:

Theorem 1.4.2 ([CH15], Theorem 3/4). *There are precisely 3 qG-deformation families of del Pezzo surfaces with $k \geq 1$ $\frac{1}{3}(1,1)$ points and Fano Index $f \geq 2$:*

1. $\mathbb{P}(1,1,3)$ *with* $k = 1$*;*
2. $X_4 \subset \mathbb{P}(1,1,1,3)$ *with* $k = 1$*;*
3. $X_6 \subset \mathbb{P}(1,1,3,3)$ *with* $k = 2$.

There are precisely 26 *qG-deformation families of del Pezzo surfaces with* $k \geq 1$ $\frac{1}{3}(1,1)$ *points and Fano Index* $f = 1$.

In their notation, let $X_{k,d}$ be the family of orbifold del Pezzo surfaces with k singularities of type $\frac{1}{3}(1,1)$ and degree $d = K_X^2$ we have:

Theorem 1.4.3 ([CH15], Theorem 6/Corollary 8). *If* $X = X_{k,d}$ *has no* (-1)*-curves passing through the singular points, then it is one of the following (with subsequent Cascade construction):*

(k=1) *X is either* $\mathbb{P}(1,1,3)$ *or* $X_4 \subset \mathbb{P}(1,1,1,3)$*; a surface in the family* $X_{1,d}$ *is either the blow-up of* $\frac{25}{3} - d$ *smooth points of* $\mathbb{P}(1,1,3)$ *or, for* $d < \frac{16}{3}$, $\frac{16}{3} - d$ *smooth points of* $X_4 \subset \mathbb{P}(1,1,1,3)$*;*

(k=2) *X is either* $X_6 \subset \mathbb{P}(1,1,3,3)$ *or* $X_{2,\frac{17}{3}}$*; a surface of the family* $X_{2,d}$ *is either the blow-up of* $\frac{17}{3} - d$ *smooth points of* $X_{2,\frac{17}{3}}$ *or, if* $d < \frac{8}{3}$, $\frac{8}{3} - d$ *smooth points of* $X_6 \subset \mathbb{P}(1,1,3,3)$*;*

(k=3) $X = X_{3,5}$*; every surface in the family* $X_{3,d}$ *is the blow-up of* $5 - d$ *smooth points of* $X_{3,5}$*;*

(k=4) $X = X_{4,\frac{7}{3}}$*; every surface in the family* $X_{4,d}$ *is the blow-up of* $\frac{7}{3} - d$ *smooth points of* $X = X_{4,7}$*;*

(k=5) $X = X_{5,\frac{5}{3}}$; *a surface in the family* $X = X_{5,\frac{2}{3}}$ *is the blow-up of a smooth point of* $X = X_{5,\frac{5}{3}}$;

(k=6) $X = X_{6,2}$; $X_{6,1}$ *is the blow-up of a smooth point of* $X_{6,2}$.

All of the above families, except for $X_{4,\frac{1}{3}}$, $X_{5,\frac{2}{3}}$ *and* $X_{6,1}$, *admit a qG-degeneration to a toric surface.*

Note that there are some smaller Cascades that can be linked to same surfaces through blow ups from different base surfaces.

1.5 Invariants

In this work, we will be considering specific singularity types: namely we will deal with cyclic quotient singularities having residual part equal to either $\frac{1}{3}(1,1)$ or $\frac{1}{5}(1,2)$. Thus, as described in Section 1.1, orbifold singularities having such R–content will be of the form:

$$\frac{1}{\omega r}(1, \omega a - 1) = \frac{1}{3(3m+1)}(1, 2(3m+1) - 1) \qquad \text{for} \qquad \frac{1}{3}(1,1) \tag{1.17}$$

$$\frac{1}{\omega r}(1, \omega a - 1) = \frac{1}{5(5m+1)}(1, 3(5m+1) - 1) \qquad \text{for} \qquad \frac{1}{5}(1,2) \tag{1.18}$$

So the basket of residual singularities for X will be of type:

$$\mathcal{B} = \{k_1 \times \tfrac{1}{3}(1,1) + k_2 \times \tfrac{1}{5}(1,2)\} \tag{1.19}$$

for $k_2 \geq 1$. As defined in 1.2.2, if n denotes the topological Euler number of X, the singularity content of the surface X is $(n, k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2))$.
Moreover, as we saw in Section 1.2, the singularity content is a qG–deformation invariant, so we can assume X has $k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2)$ singularities.
The Hirzebruch–Jung continued fractions for these singularities are

$$\frac{3}{1} = [3] \quad \text{and} \quad \frac{5}{2} = [3,2].$$

From Orbifold Riemann-Roch (as described by 1.5), we have:

$$h^0(X, -K_X) = 1 + K_X^2 - \frac{1}{3}k_1 - \frac{2}{5}k_2. \tag{1.20}$$

Recall that if $\varphi : Y \to X$ is the minimal resolution of X, then over every singularity we have a chain of curves where the self–intersection are given by the coefficients of the Hirzebruch–Jung continued fractions. Thus we have

$$\rho(Y) = \rho(X) + k_1 + 2k_2 \tag{1.21}$$

As the lengths of the continued fractions are 1 and 2 respectively for the two types of singularities, we have

$$\begin{aligned} n &= \rho(Y) + 2 - 2k_1 - 3k_2 = \\ &= \rho(X) + 2 - k_1 - k_2 \end{aligned} \tag{1.22}$$

Moreover, by the Noether formula introduced in 1.8 we have

$$\begin{aligned} K_X^2 &= 12 - n - \frac{5}{3}k_1 - \frac{13}{5}k_2 = \\ &= 10 - \rho(X) - \frac{2}{3}k_1 - \frac{8}{5}k_2 \end{aligned} \tag{1.23}$$

As we are dealing with del Pezzo surfaces, then necessarily $K_X^2 > 0$. Moreover, $h^0(X, -K_X) \geq 0$, but for now we will assume $h^0(X, -K_X) > 0$. So, putting together these two conditions with $n \geq 0$, we have the system of equations:

$$\begin{cases} k_1 + k_2 \leq \rho(X) + 2 \\ \rho(X) \leq 10 - k_1 - 2k_2 \end{cases} \tag{1.24}$$

The integer solutions of such system are listed in the Table 1.1 below.

k_1	0	1	2	3	4	0	1	2	3	0	1	0
k_2	1	1	1	1	1	2	2	2	2	3	3	4
$\rho(X)$	1..8	1..7	1..6	2..5	$3,4$	1..6	1..5	$2,3,4$	3	1..4	$2,3$	2
n	2..9	1..7	0..5	0..3	$0,1$	1..6	0..4	$0,1,2$	0	0..3	$0,1$	0

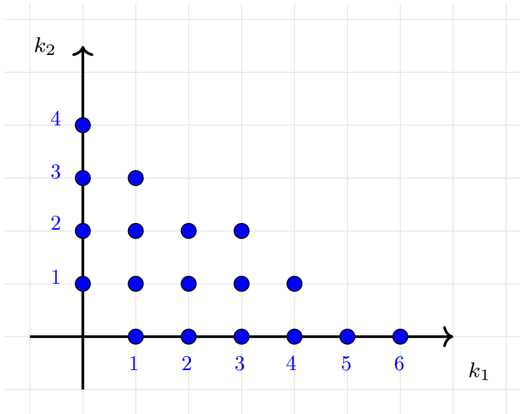
Table 1.1: Invariants for Orbifold del Pezzo surfaces with $h^0(-K_X) \neq 0$

Thus, if a orbifold del Pezzo surface X has singularity content $(n, \mathcal{B})$ where $\mathcal{B}$ denotes the basket of singularities as defined above, then X falls in one of the listed numerical cases.

Every column represents the ranges of the Picard rank and of the topological Euler number for every choice of available pair (k_1, k_2).

Definition 1.5.1. If X is a del Pezzo orbifold with singularity content $(n, k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2))$, then we say the surface X is of **type** (k_1, k_2).

The singularity type is thus given by the pairs (k_1, k_2), and the possible pairs are represented in the following diagram:



We know that for every pair (k_1, k_2) there are numerical candidates for these surfaces, so in the next Chapter we will check which of such candidates actually exist.

Chapter 2

Minimal Surfaces

In this chapter we recall some notions about the Minimal Model Program for surfaces and extend it to orbifold del Pezzo surfaces; we then proceed with case analyses to find model surfaces of orbifolds with the specified singularities.

2.1 Mori Theory and Minimal Surfaces

In this section we recall some of the main results of the Mori Program for smooth surfaces, for details about theorems and proofs see [And1], [Reid93] and [BCHM].

Let X be a smooth complex projective variety. It is known that there exists a well defined pairing :

$$\begin{aligned} \mathrm{Pic}(X) \times Z_1(X) &\to \mathbb{Z} \\ (\quad L, \quad C \quad) &\mapsto L \cdot C := deg(L\mid_C) \end{aligned} \tag{2.1}$$

where Pic(X) is the Picard group of X, $Z_1(X)$ is the group of 1-cycles of X, $L \in$ Pic(X) line bundle and $C \subset X$ an irreducible reduced curve. We say two line bundles L_1, L_2 are numerically equivalent ($L_1 \overset{\text{num}}{\sim} L_2$) if $L_1 \cdot C = L_2 \cdot C$ for every curve $C \subset X$ (and dually we define the linear equivalence for the space of 1-cycles).This definition makes the spaces:

$$N^1(X) = (\mathrm{Pic}(X)/ \overset{\text{num}}{\sim}\ ) \otimes \mathbb{R}$$
$$N_1(X) = (\mathrm{Z}_1/ \overset{\text{num}}{\sim}\ ) \otimes \mathbb{R}$$

into dual $\mathbb{R}$-vector spaces, where $N^1(X)$ is the vector space of linear forms on $N_1(X)$.

In $N_1(X)$, we define the cone of effective 1-cycles as follows:

$$\mathrm{NE}(X) = \{C \in N_1(X) \ \mid \ C = \sum r_i C_i,\ r_i \in \mathbb{R}_{\geq 0},\ C_i \in N_1(X)\ \mathit{irreducible}\}.$$

Its closure $\overline{\mathrm{NE}}(X) \subset N_1(X)$ with respect to the real topology is called the *Mori cone*. We have the following:

Theorem 2.1.1 (Kleiman criterion)**.** *If D is a $\mathbb{Q}$-divisor class in* $\mathrm{Pic}(X) \otimes \mathbb{Q}$*, then*

$$D \text{ is ample } \Leftrightarrow D \cdot Z > 0 \ \forall Z \in \overline{\mathrm{NE}}(X) \setminus \{0\}$$

Therefore, the set of ample divisors represents the interior part of the nef cone in $N_1(X)$. In particular, we have a link between nef and ample divisors:

Corollary 2.1.1. *If $D, H \in \mathrm{Pic}(X)$, D nef and H ample, then $D^2 \geq 0$ and $D + \lambda H$ is ample for all $\lambda \in \mathbb{Q}_{>0}$.*

For every closed convex cone $V \subset \mathbb{R}$ we have the following standard definition:

Definition 2.1.1. A subcone $W \subset V$ is called ***extremal*** if

$$u, v \in V,\ u + v \in W \Rightarrow u, v \in W.$$

Every extremal 1-dimensional cone is called ***extremal ray***.

By using the geometry of the nef cone, we obtain a good description of curves in $\overline{NE}(X)$:

Propostion 2.1.1. *Let C be irreducible curve on X, then:*

1. $C^2 \leq 0 \Rightarrow [C]$ *is in the boundary of* $\overline{\mathrm{NE}}(X)$*;*
2. $C^2 < 0 \Rightarrow [C]$ *is extremal in* $\overline{\mathrm{NE}}(X)$*.*

On the other hand, we have the following characterisation of extremal curves:

Propostion 2.1.2. *If $[D] \in \overline{NE}(X)$ is extremal, then either $D^2 \leq 0$, or $\rho(X) = 1$. Moreover, if $D^2 < 0$, then the extremal ray is spanned by the class of an irreducible curve.*

As a result, if $\mathbb{R}_+[l]$ is a class of an extremal ray, then l is one of the following:

- (-1)-curve
- fibre of $\mathbb{P}^1$-bundle
- a line in $\mathbb{P}^2$

The idea of the Minimal Model Program is to assign a well understood model to a lesser known variety by means of birational maps. We start with a variety whose canonical divisor is not nef, and by subsequent birational maps we would like to get a model $\overline{X}$ for which $K_{\overline{X}}$ is nef. In particular, the fact that K_X is not nef can be interpreted in terms of the nef cone in the following way: if we consider $K_X \in N^1(X)$ as a form on $N_1(X)$, then the hyperplane defined by all the elements $C \in N_1(X)$ such that $K_X \cdot C = 0$ cuts $N_1(X)$ in half–spaces.

Theorem 2.1.2 (Rationality Theorem). *Let X be a smooth surface such that K_X is not nef. Let H be an ample class on X, then there exists a nef threshold defined by:*

$$\mu = \sup\{t \in \mathbb{R} \mid tK_X + H \text{ is nef}\}$$

such that $\mu \in \mathbb{Q}$ and either μ, 2μ or $3\mu \in \mathbb{Z}$.

As a result, the hyperplane defined by the linear form K_X divides the nef cone in two parts contained in the half-spaces of $N_1(X)$ containing curves that have either positive or negative intersection with K_X. We will denote the two half–spaces with $K_X \geq 0$ and $K_X \leq 0$ respectively, and similarly the two parts of the nef cone contained in such half–spaces with $\overline{\mathrm{NE}}(X)_{K_X \geq 0}$ and $\overline{\mathrm{NE}}(X)_{K_X \leq 0}$.

In particular, the part of the cone contained in the negative halfspace is locally polyhedral and spanned by a countable number of classes of extremal rays.

Theorem 2.1.3 (Cone Theorem). *Let X be a smooth surface, H an ample divisor on X and R_i the extremal rays of $\overline{\mathrm{NE}}(X)$. Then*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum R_i$$

and for every $\epsilon > 0$ there is only a finite number of R_i such that $(K_X + \epsilon H) \cdot R_i \leq 0$.

Remark 2.1.1. It follows from the results above that if a smooth surface X has an extremal ray $R_i = [C]$ such that $C^2 > 0$, then $X = \mathbb{P}^2$. Moreover, from the birational classification of del Pezzo surfaces every smooth del Pezzo is either a blow up of $\mathbb{P}^2$ in $d < 9$ general points or $\mathbb{P}^1 \times \mathbb{P}^1$. As a consequence, every del Pezzo surface has a finite number of extremal rays R_i and these generate the K_X negative part of the nef cone, i.e. the part of cone contained in the half–space $K_X < 0$.

Now, to every extremal ray R_i of $\overline{NE}(X)$ we can associate a contraction, namely an extremal contraction, that is a morphism of X to a projective variety that contracts all the curves in the class of R_i:

Definition 2.1.2. Let R be an extremal ray of $\overline{NE}(X)$. A projective morphsm $\psi : X \to X'$ between normal projective varieties is called ***extremal contraction*** if the following conditions hold:

1. Every irreducible curve $C \subset X$ is contracted to a point if and only if C is in the class of R;
2. ψ has connected fibres;
3. If $A \in Div(X')$ is an ample class of X', then $H = \psi^*(A)$ is nef on $Pic(X)$

The possible outcomes of this contraction can be classified. Indeed, we have the following:

Theorem 2.1.4 (Contraction Theorem). *Let R be an extremal ray in $\overline{NE}(X)_{K_X<0}$ and $\psi_R = \psi : X \to Z$ is the associated extremal contraction. Then we are in one of the following cases:*

1. *Z is a smooth surface and ψ represents a blow-up of Z in a smooth point and $\rho(Z) = \rho(X) - 1$;*
2. *Z is a smooth curve and X is a (minimal) ruled surface over Z;*
3. *Z is a point and X is a del Pezzo surface of rank 1, i.e. $X \cong \mathbb{P}^2$.*

So, given any smooth surface, it is possible to apply the contraction theorem finitely many times to obtain a surface that would represent a model for our original variety.

Definition 2.1.3. A variety X is called ***Minimal Model*** if K_X is nef and it has only $\mathbb{Q}$-factorial terminal singularities.
It is called a ***Mori Fibre Space*** if X is smooth and $\psi : \ X \to Z$ is an extremal contraction of an extremal ray with $\dim X > \dim Z$.

For surfaces, terminal singularities are smooth points. Thus, the contraction theorem ensures that, after finitely many steps, we have a model surface that is either a minimal model or a Mori Fibre Space.

Theorem 2.1.5 (Minimal Model Program of smooth surfaces)**.** *Let X be a smooth surface, then after a finite number of extremal contractions we have a birational map $\psi : X \dashrightarrow X'$ such that X' is smooth and it falls in one of the following cases:*

1. *X' is a Minimal Model;*
2. *X' is a Mori Fibre Space.*

Thus, by putting together 2.1.4 and 2.1.5 we have:

Theorem 2.1.6. *Every proper birational morphism between nonsingular projective surfaces can be factored into a sequence of blow-ups at smooth points; in other words, for any projective morphism $\psi : \ X \to Z$ there exists a sequence of contractions of (-1)-curves $\psi_i : \ X_i \to X_{i+1}$ for $i = 0, \ldots, m$ with $X_0 = X$ and $X_m = Z$.*

From the properties listed above, we reach the following result about the negativity of the exceptional locus for projective morphisms:

Theorem 2.1.7. *Let $\{\Gamma_i\}$ a finite set of curves on a nonsigular surface X contracted to points by a projective birational morphism $\psi : \ X \to Z$, then the quadratic form defined by the matrix $(\Gamma_i \cdot \Gamma_j)$ is negative definite.*

Finally, we link this construction to the *Kodaira dimension*, a birational invariant strictly intertwined with the Enriques–Kodaira classification of algebraic surfaces (see [Reid93]).

Definition 2.1.4. Let X be a smooth projective surface. The **Kodaira dimension** $k(X)$ of X is defined to be

$$\begin{aligned} k(X) = -\infty \qquad \text{if } H^0(X, \mathcal{O}_X(mK_X)) = 0 \quad \text{for all } m \in \mathbb{N} \\ k(X) = (\text{trasdeg}_{\mathbb{C}} \bigoplus_m H^0(X, \mathcal{O}_X(mK_X))) - 1 \\ \text{if } H^0(X, OX(mK_X)) \neq 0 \quad \text{for some } m \in \mathbb{N} \end{aligned} \tag{2.2}$$

Theorem 2.1.8. *Let X be a smooth projective surface, then the outcome of an MMP for X is a Minimal Model if and only if $k(X) \geq 0$ and a Mori Fibre Space if and only if $k(X) = -\infty$.*

It is well–known that a smooth del Pezzo surfaces is birationally equivalent to either $\mathbb{P}^2$ blown up in $9-d$ general points (with $0 \leq d \leq 8$) or $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, running an MMP on smooth rational surfaces will end in either $\mathbb{P}^2$ or a Hirzebruch surface, i.e. ruled surfaces over $\mathbb{P}^1$. In the case of $\mathbb{Q}$–Gorenstein surfaces we are still lacking a complete classification, and this makes a generalisation of this construction reasonably complicated. We thus focus on a specific pool of cases with the aim of finding a sensible way to include broader classes of singularities.

To this end, we will relate the extremal contractions available on the singular surfaces to the smooth case via their minimal resolutions. Thus, the first step is to understand what the possible outcomes of the MMP are, in order to identify the surfaces that in the unprojection contruction (as described in 1.3.1) will be at the top of the cascade of blow ups. Roughly speaking, this will require the surface to be "minimal" in terms of dimension of the Picard group. This motivates the following definition:

Definition 2.1.5. A (-1)-curve is said to be **floating** if it is entirely contained in the smooth locus of X. The surface X is said to be **minimal** if it has no floating (-1)-curves.

2.2 Minimal Surfaces with $\rho = 1$

In this section we discuss possible surfaces that can arise as an endpoint of an MMP for our class of log del Pezzo surfaces. As we will see, this will give an analogue result to the list in [CH15], but with reasonably more cases.

2.2.1 Model for $\rho(X) = 1$ and $\mathcal{B} = \{\frac{1}{5}(1,2)\}$

We will first focus on the case X has Picard rank 1 and admits just one singularity of type $\frac{1}{5}(1,2)$.

Recall from Section 1.1 that the Orbifold Riemann–Roch formula (1.5) and the Ice cream functions (1.4) give us information about the invariants of our varieties.
Indeed, for the pluricanonical genus we have:

$$h^0(-mK_X) = 1 + \frac{m(m+1)}{2}K_X^2 + c_{\frac{1}{5}(1,2)}(-mK_X)$$

where $c_{\frac{1}{5}(1,2)}(-mK_X)$ are the contributions of the quotient singularity $\frac{1}{5}(2,4)$.

Thus, by plugging in the contribution for the specific basket we get:

$$h^0(-mK_X) = 1 + \frac{m(m+1)}{2}\left(12 - n - \frac{13}{5}\right) + \begin{cases} 0 & m \equiv 4,5 \\ -\frac{2}{5} & m \equiv 1,3 \\ -\frac{1}{5} & m \equiv 2 \end{cases} \tag{2.3}$$

Moreover, as we are assuming $\rho(X) = 1$, the topological Euler number (as defined in 1.22) is $n = \rho(X) + 1 = 2$. Thus the Hilbert series for the surface X is given by:

$$P_{-K_X}(t) = 1 + 8t + 23t^2 + 45t^3 + 75t^4 + 112t^5 + 156t^6 + 208t^7 + 267t^8 + \cdots \tag{2.4}$$

By including the generators for the orbinates of $\frac{1}{5}(1,2)$ we have the multiplied Hilbert series:

$$P_{-K_X}(t)(1-t)(1-t^2)(1-t^5) = 1 + 7t + 14t^2 + 15t^3 + 15t^4 + 14t^5 + 7t^6 + t^7$$

Giving for example the multiplied series:

$$\begin{aligned} P_{-K_X}(t) \prod_{a \in [1,2,2,3,4,5]} (1-t^a) = 1 + 7t + 13t^2 + 7t^3 - 7t^4 - 21t^5 - 29t^6 - 21t^7 \\ - 7t^8 + 7t^9 + 13t^{10} + 7t^{11} + t^{12}. \end{aligned}$$

related to a possible embedding in $\mathbb{P}(1,2,2,3,4,5)$.
Thus, consider the surface X with minimal resolution $\varphi : Y \to X$. Then

$$K_Y = \varphi^*(K_X) - \tfrac{1}{5}(2C_1 + C_2)$$

where C_1, C_2 are the exceptional curves coming from the resolution, such that $(C_1)^2 = -3$ and $(C_2)^2 = -2$. As $\rho(X) = 1$, then $\rho(Y) = 3$, so Y is not minimal. Consequently, there must exists an extremal curve L such that $L^2 = -1$ and that intersects the exceptional curves in $L \cdot C_1 = a$ and $L \cdot C_2 = b$ ($a, b \in \mathbb{N}$) points respectively. As Y is smooth, then $NS(Y) = \mathrm{Pic}(Y) = H^2(Y, \mathbb{Z})$ and by Poincaré duality we have the perfect pairing 2.1 is unimodular (as X has no torsion). Thus, as the triple (C_1, C_2, L) spans a sublattice of $H^2(Y, \mathbb{Z})$ of rank 3, the matrix M representing the pairing in this basis must have $\det(M) = \pm d^2$.
On the other hand we have

$$1 = -K_Y \cdot L = -K_X \cdot L + \tfrac{1}{5}(2a + b)$$

which gives

$$2a + b < 5. \tag{2.5}$$

Morever, if we consider the intersection matrix M for the basis (C_1, C_2, L), we see that none of the solutions to the inequality 2.5 satisfy $det(M) = \pm d^2$, so no such surface X exists.

2.2.2 Towards the general case

Consider now a del Pezzo surface X with

$$\mathrm{Sing}(X) = \{k_1 \times \frac{1}{3}(1,1), k_2 \times \frac{1}{5}(1,2), n_1 \times A_1, n_2 \times A_2, n_3 \times A_3, n_4 \times A_4\} \tag{2.6}$$

As we have seen in Section 1.5, if $\varphi : Y \to X$ is the minimal resolution of X, then over every singularity of type $\frac{1}{3}(1,1)$ there is exactly one exceptional curve and for $\frac{1}{5}(1,2)$ there are two. Moreover, over every singularity of type $A_m = \frac{1}{m+1}(1,m)$ there is a chain of m (-2)–curves intersecting transversely. As $\rho(Y) \leq 11$, then the set 2.6 is such that $k_1 + 2k_2 + n_1 + 2n_2 + 3n_3 + 4n_4 \leq 11$. Notice we are also including the case some of the listed integers is 0. Assume now $\rho(X) = 1$, then by Belusov's theorem we have:

Theorem 2.2.1 ([Bel09], Theorem 1.1)**.** *Let X be a log del Pezzo surface with $\rho(X) = 1$. Then $|Sing(X)| \leq 4$.*

Therefore we have:

$$k_1 + k_2 + n_1 + n_2 + n_3 + n_4 \leq 4 \tag{2.7}$$

Let r denote the number of exceptional curves appearing in the minimal resolution Y of X:

- E_i denote the (-3)-curves from the $\frac{1}{3}(1,1)$ singularities;

- C_1^i, C_2^i be the (-3) and (-2) curves (respectively) from the $\frac{1}{5}(1,2)$ singularities;
- F_m^i be the (-2)-curves from the A_m singularities.

As mentioned in 1.7, for our singular locus we have:

$$K_Y = \varphi^*(K_X) - \tfrac{1}{3}\sum_i E_i - \tfrac{1}{5}\sum_j (2C_1^j + C_2^j) \tag{2.8}$$

Since $\rho(X) = 1$, then $H^2(Y, \mathbb{Z})$ admits a sublattice of rank $r+1$ generated by

$$\begin{aligned}\langle K_X, \{E_i\}_{i=1..k_1}, &\{C_1^i, C_2^i\}_{i=1..k_2}, \{F_1^i\}_{i=1..n_1}, \{F_{2,1}^i, F_{2,2}^i\}_{i=1..n_2},\\ &\{F_{3,1}^i, F_{3,2}^i, F_{3,3}^i\}_{i=1..n_3}, \{F_{4,1}^i, F_{4,2}^i, F_{4,3}^i, F_{4,4}^i\}_{i=1..n_4}\rangle\end{aligned} \tag{2.9}$$

Notice the mild abuse of notation: as we are including the possibility that $k_i, n_i = 0$ for some i, then some of the listed curves would not appear in the set 2.9. Thus we are including the curves in the respective subset if $k_i, n_i \neq 0$.
As the curves coming from resolution of two distinct singularities are disjoint, the intersection matrix has the form of a block diagonal matrix: the first block is the element $(K_X)^2 = (\varphi^*(K_X))^2$ and the remaining blocks are given by the intersection pairing of the configurations of curves coming from the individual singularities.
As a result, the determinant of the total intersection matrix M is:

$$\det(M) = K_X^2(-3)^{k_1}(5)^{k_2}(-2)^{n_1}(3)^{n_2}(-4)^{n_3}(5)^{n_4}$$

and consequently

$$|\det(M)| = (9 - n_1 - 2n_2 - 3n_3 - 4n_4 - \tfrac{2}{3}k_1 - \tfrac{8}{5}k_2)2^{n_1+2n_3}3^{n_2}5^{n_4} \tag{2.10}$$

Similarly as Section 2.2.1, we look for solutions of $|\det(M)|$ being a perfect square; so the cases that satisfy this condition give the following possibilities for $\mathrm{Sing}(X)$:

$$\begin{array}{cccc} A_1 & A_1 + A_2 & 3 \times A_2 & 4 \times A_2 \\ A_4 & 2 \times A_3 & 2 \times A_1 + A_3 & 2 \times A_2 + A_3 + \frac{1}{3}(1,1) \\ \frac{1}{3}(1,1) & 2 \times A_4 & A_2 + A_3 + \frac{1}{5}(1,2) & 2 \times A_1 + 2 \times A_3 \\ & A_1 + \frac{1}{5}(1,2) & A_1 + 2 \times A_3 & A_1 + 3 \times \frac{1}{3}(1,1) \\ & A_3 + \frac{1}{3}(1,1) & A_1 + A_4 + \frac{1}{3}(1,1) & A_1 + 2 \times A_3 + \frac{1}{5}(1,2) \\ & A_2 + \frac{1}{5}(1,2) & 2 \times A_4 + \frac{1}{3}(1,1) & \\ & & A_4 + 2 \times \frac{1}{5}(1,2) & \end{array} \tag{2.11}$$

In the same fashion as the example with $\mathrm{Sing}(X) = \frac{1}{5}(1,2)$ above (i.e. by considering an extremal ray $L \in Y$ and its intersections with the exceptional curves) we can eliminate some more cases. Indeed, if the surface X admits $2 \times A_3$, or $2 \times A_4 + \frac{1}{3}(1,1)$, or $2 \times A_2 + A_3 + \frac{1}{3}(1,1)$, or $A_1 + 3 \times \frac{1}{3}(1,1)$ singularities, then the intersection of the curve L with K_X is nonnegative, contradicting X being a del Pezzo orbifold.

Ultimately, we have:

Lemma 2.2.1. *Let X be a del Pezzo surface with Picard rank $\rho(X) = 1$ and*

$$\mathrm{Sing}(X) = \{k_1 \times \frac{1}{3}(1,1), k_2 \times \frac{1}{5}(1,2), n_1 \times A_1, n_2 \times A_2, n_3 \times A_3, n_4 \times A_4\}$$

such that $k_1 + 2k_2 + n_1 + 2n_2 + 3n_3 + 4n_4 \leq 11$. Then $\mathrm{Sing}(X)$ is one of the listed cases in Table 2.1 below.

n°singularities	Type
1	$\{A_1\}$, $\{A_4\}$, $\frac{1}{3}(1,1)$
2	$\{A_1 + A_2\}$, $\{2 \times A_4\}$, $\{A_1 + \frac{1}{5}(1,2)\}$, $\{A_2 + \frac{1}{5}(1,2)\}$, $\{A_3 + \frac{1}{3}(1,1)\}$
3	$\{A_1 + 2 \times A_3\}$, $\{2 \times A_1 + A_3\}$, $\{3 \times A_2\}$, $\{A_2 + A_3 + \frac{1}{5}(1,2)\}$, $\{A_4 + 2 \times \frac{1}{5}(1,2)\}$, $\{A_1 + A_4 + \frac{1}{3}(1,1)\}$
4	$\{2 \times A_1 + 2 \times A_3\}$, $\{4 \times A_2\}$, $\{A_1 + 2 \times A_3 + \frac{1}{5}(1,2)\}$

Table 2.1: Possible singularities for X log del Pezzo with $\rho(X) = 1$

2.3 Minimal Model Program for surfaces of type (k_1, k_2)

Let X be a del Pezzo surface with cyclic quotient singularities, and let Y be its minimal resolution. Recall that over every singular point the exceptional locus coming from the map $\varphi : Y \to X$ consists of a chain of smooth rational curves with negative self intersection (two adjacent curves meet transversely in one point). If Γ is an irreducible rational curve passing through a bunch of singular points of X, then its birational transform $\widetilde{\Gamma}$ via φ in Y meets transversely exactly one curve of the exceptional locus over each point.

Theorem 2.3.1 (Directed MMP). *Let X be del Pezzo surface with $h^0(X, -K_X) \neq 0$ and singular locus given by*

$$\mathrm{Sing}(X) = \{k_1 \times \frac{1}{3}(1,1), k_2 \times \frac{1}{5}(1,2), n_1 \times A_1, n_2 \times A_2, n_3 \times A_3, n_4 \times A_4\} \tag{2.12}$$

singularities such that $k_1 + 2k_2 + n_1 + 2n_2 + 3n_3 + 4n_4 \leq 11$. Let $\varphi : Y \to X$ be the minimal resolution of the surface X such that:

- *$F^i_m \subset Y$ $i = 1, \dots, m$ are the (-2)-curves coming from A_m singularities;*
- *$E_i \subset Y$ are the (-3)-curves coming from the $\frac{1}{3}(1,1)$ singularities;*
- *$C^i_1, C^i_2 \subset Y$ are the (respectively) $(-3), (-2)$-curves from the $\frac{1}{5}(1,2)$ singularities*

Let $\psi_i : (X_{i-1}, \Gamma) \to (X_i, P)$ be the extremal contraction corresponding to the curve Γ. The proper transform $\widetilde{\Gamma} \subset Y$ of Γ is a (-1)-curve meeting transversely at most one exceptional curve of $\varphi_{i-1} : Y_{i-1} \to X_{i-1}$ above each singularity. Let Q_1, Q_2 be singular points on X and B_1, B_2 be two exceptional curves in the configuration of the resolution of the points Q_1, Q_2 respectively, such that if Γ passes through Q_1, Q_2 then it intersects the curves B_1, B_2. Then for an extremal contraction one of the following holds:

($\mathcal{C}$) X_i is a del Pezzo surface with $\rho(X_i) = \rho(X_{i-1}) - 1$, ψ_i is a divisorial contraction of the curve $\Gamma \in X_{i-1}$ to a point P. Then the following table summarises the possible cases for the contraction ψ_i:

	Q_1	Q_2	P	$K_X \cdot \Gamma$	Γ^2	B_1	B_2
$\mathcal{C}0$			$*$	-1	-1		
$\mathcal{C}1$	A_1		$*$	-1	$-\frac{1}{2}$	F	
$\mathcal{C}2$	A_2		$*$	-1	$-\frac{1}{3}$	F_1	
$\mathcal{C}3$	A_3		$*$	-1	$-\frac{1}{4}$	F_1	
$\mathcal{C}4$	A_4		$*$	-1	$-\frac{1}{5}$	F_1	
$\mathcal{C}5$	$\frac{1}{5}(1,2)$		A_1	$-\frac{4}{5}$	$-\frac{2}{5}$	C_2	
$\mathcal{C}6$	$\frac{1}{3}(1,1)$		A_1	$-\frac{2}{3}$	$-\frac{2}{3}$	E	
$\mathcal{C}7$	A_1	$\frac{1}{3}(1,1)$	$*$	$-\frac{2}{3}$	$-\frac{1}{6}$	F	E
$\mathcal{C}8$	$\frac{1}{5}(1,2)$		A_2	$-\frac{3}{5}$	$-\frac{3}{5}$	C_1	
$\mathcal{C}9$	A_1	$\frac{1}{5}(1,2)$	$*$	$-\frac{3}{5}$	$-\frac{1}{10}$	F	C_1
$\mathcal{C}10$	$\frac{1}{3}(1,1)$	$\frac{1}{5}(1,2)$	$*$	$-\frac{7}{15}$	$-\frac{1}{15}$	E	C_2
$\mathcal{C}11$	$\frac{1}{3}(1,1)$	$\frac{1}{3}(1,1)$	A_2	$-\frac{1}{3}$	$-\frac{1}{3}$	E_1	E_2
$\mathcal{C}12$	$\frac{1}{3}(1,1)$	$\frac{1}{5}(1,2)$	A_3	$-\frac{4}{15}$	$-\frac{4}{15}$	E	C_1
$\mathcal{C}13$	$\frac{1}{5}(1,2)$	$\frac{1}{5}(1,2)$	A_4	$-\frac{1}{5}$	$-\frac{1}{5}$	C_1^1	C_1^2

Table 2.2: Divisorial Contraction

($\mathcal{F}$) $X_i = \mathbb{P}^1$, i.e. ψ_i is a Mori Fibre Space. If Γ is the special fibre on X, then the following table summarises the possible cases for the fibration ψ_i:

	Q_1	Q_2	$K_X \cdot \Gamma$	Γ^2	B_1	B_2
$\mathcal{F}0$	A_3		-1	0	F_2	
$\mathcal{F}1$	A_1	A_1	-1	0	F^1	F^2
$\mathcal{F}2$	A_2	$\frac{1}{3}(1,1)$	$-\frac{2}{3}$	0	F_1	E
$\mathcal{F}3$	$\frac{1}{5}(1,2)$	$\frac{1}{5}(1,2)$	$-\frac{2}{5}$	0	C_1^1	C_2^2

Table 2.3: Special Fibres for Mori Fibre Spaces

($\mathcal{M}$) $X_i = \{pt\}$ is a point; thus X_{i-1} is a del Pezzo surface of rank $\rho(X_{i-1}) = 1$ and it is one of the following:

	Surface	Singularities
$\mathcal{M}1$	$\mathbb{P}^2$	
$\mathcal{M}2$	$\mathbb{P}(1,1,2)$	A_1
$\mathcal{M}3$	$\mathbb{P}(1,2,3)$	$A_1 + A_2$
$\mathcal{M}4$	$\mathbb{P}^2/\mu_3$	$3 \times A_2$
$\mathcal{M}5$	$\mathbb{P}(1,1,3)$	$\frac{1}{3}(1,1)$
$\mathcal{M}6$	$\mathbb{P}(1,3,4)$	$A_3 + \frac{1}{3}(1,1)$
$\mathcal{M}7$	$\mathbb{P}(1,2,5)$	$A_1 + \frac{1}{5}(1,2)$
$\mathcal{M}8$	$\mathbb{P}(1,3,5)$	$A_2 + \frac{1}{5}(1,2)$
$\mathcal{M}9$	$\mathbb{P}(2,3,5)$	$A_1 + A_4 + \frac{1}{3}(1,1)$
$\mathcal{M}10$	$\mathbb{P}(3,4,5)$	$A_2 + A_3 + \frac{1}{5}(1,2)$
$\mathcal{M}11$	$M_{2,2} \subset \mathbb{P}^4$	$2 \times A_1 + A_3$
$\mathcal{M}12$	$N_8 \subset \mathbb{P}(1,1,2,5)$	A_4
$\mathcal{M}13$	$X^{5,3}_{(0,2)}$	$A_4 + 2 \times \frac{1}{5}(1,2)$

Table 2.4: Del Pezzo Surfaces with $\rho(X) = 1$

Proof. Let X be a del Pezzo surface with basket $\mathcal{B}$ as above, and let $\psi : X \to \bar{X}$ a proper birational morphism. Consider then the minimal resolutions $\varphi : Y \to X$ and $\bar{\varphi} : \bar{Y} \to \bar{X}$. The by minimality of the resolutions, the morphism ψ lifts to a morphism $\bar{\psi}$ such that we have the commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\bar{\psi}} & \bar{Y} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\bar{\varphi}} \\ X & \xrightarrow{\psi} & \bar{X} \end{array}$$

Then, by Theorem 2.1.6, the morphism $\bar{\psi}$ can be factored into a sequence of ordinary blow ups. Suppose Y and $\bar{Y}$ are not isomorphic; then as they are both nonsingular, there must exist a (-1)-curve $\widetilde{\Gamma}$ such that $\bar{\varphi} \circ \bar{\psi}(\widetilde{\Gamma}) = \{p\}$, for p point in $\bar{X}$. Let Θ be a set of curves such that $\Theta = \bar{\psi}^{-1}(\bar{\varphi}^{-1}(p))$, and let D be the exceptional locus of φ.
Then, if $\widetilde{\Gamma} \cdot D = \emptyset$, then $\widetilde{\Gamma}$ is not exceptional for φ and its birational transform is a (-1)-curve on X. By applying Castelnuovo's criterion we obtain a morphism $\psi' : X \to X_1$ corresponding to the blow up of a smooth point with exceptional locus given by Γ. Thus ψ factors through ψ' and we obtain a morphism $X_1 \to \bar{X}$.

If $\widetilde{\Gamma} \circ D \neq \emptyset$, then there exists a copmponent D_0 of D such that $D_0 \circ \widetilde{\Gamma} \neq \emptyset$. So D_0 is contracted by φ and $\psi(\varphi(D_0)) \subset \psi(\varphi(\widetilde{\Gamma})) = p$, so $D_0 \subset \Theta$. As D_0 is in the exceptional locus of X, then $(D_0)^2 = -2$ or -3, hence, as Θ is negative definite by 2.1.7, we obtain a list of possible cases by considering the intersections of the birational transform $\Gamma = \varphi(\widetilde{\Gamma})$ in the table below.

Singularities	Γ^2	$K_X \cdot \Gamma$
$\emptyset$	-1	-1
A_1	$-\frac{1}{2}$	-1
A_2	$-\frac{1}{3}$	-1
A_3 (I)	$-\frac{1}{4}$	-1
A_3 (II)	0	-1
A_4 (I)	$-\frac{1}{5}$	-1
A_4 (II)	$\frac{1}{5}$	-1
$\frac{1}{3}(1,1)$	$-\frac{2}{3}$	$-\frac{2}{3}$
$\frac{1}{5}(1,2)$ (I)	$-\frac{2}{5}$	$-\frac{4}{5}$
$\frac{1}{5}(1,2)$ (II)	$-\frac{3}{5}$	$-\frac{3}{5}$
$A_1 + \frac{1}{3}(1,1)$	$-\frac{1}{6}$	$-\frac{2}{3}$
$A_1 + \frac{1}{5}(1,2)$ (I)	$-\frac{1}{10}$	$-\frac{3}{5}$
$A_1 + \frac{1}{5}(1,2)$ (II)	$\frac{1}{10}$	$-\frac{4}{5}$
$2 \times \frac{1}{3}(1,1)$	$-\frac{1}{3}$	$-\frac{1}{3}$
$\frac{1}{3}(1,1) + \frac{1}{5}(1,2)$ (I)	$-\frac{1}{15}$	$-\frac{7}{15}$
$\frac{1}{3}(1,1) + \frac{1}{5}(1,2)$ (II)	$-\frac{4}{15}$	$-\frac{4}{15}$
$2 \times \frac{1}{5}(1,2)$ (I)	$-\frac{1}{5}$	$-\frac{1}{5}$
$2 \times \frac{1}{5}(1,2)$ (II)	$\frac{1}{5}$	$-\frac{3}{5}$
$2 \times \frac{1}{5}(1,2)$ (III)	0	$-\frac{2}{5}$
$2 \times A_1$	0	-1
$A_2 + \frac{1}{3}(1,1)$	0	$-\frac{2}{3}$

All the remaining cases with Γ passing through 2 or more singularities either have $\Gamma^2 > 0$, contradicting the negative definitess of Θ, or $K_X \cdot \Gamma > 0$, contradicting the ampleness of $-K_X$.

Thus, by excluding these cases, we obtain the list of contractions described in Tables 2.2 and 2.3.

From these lists, it is clear that if $\psi_i : X_i \to X_{i-1}$ is one of the contractions listed, then

the singularity type of X_{i-1} will still be of type 2.12.
Moreover, if $\widetilde{\Gamma}$ is not a φ-exceptional curve, and $\widetilde{\Gamma}^2 = -d$ with $d \neq 2, 3$, then

$$\begin{aligned} K_Y \cdot \widetilde{\Gamma} &= \left(\varphi^* K_X - \tfrac{1}{3} \sum_i E_i - \tfrac{1}{5} \sum_j \left(2C_j^1 + C_j^2 \right) \right) \cdot \widetilde{\Gamma} = \\ &= K_X \cdot \varphi_* \widetilde{\Gamma} - \tfrac{1}{3} \sum_i E_i \cdot \widetilde{\Gamma} - \tfrac{1}{5} \sum_j \left(2C_j^1 + C_j^2 \right) \cdot \widetilde{\Gamma} < 0 \end{aligned} \tag{2.13}$$

thus $d + 2g - 2 < 0$ implies $g = 0$ and $d = 1$. Hence $\widetilde{\Gamma}$ is a rational curve with $\widetilde{\Gamma}^2 = -1$.
Lastly, by induction, we can show that every X_i in the chain of contractions is a orbifold del Pezzo surface: indeed, $X = X_0$ is a orbifold del Pezzo by hypothesis. So suppose X_{i-1} is a del Pezzo with singular locus as in 2.12, $\psi_i : X_{i-1} \to X_i$ a divisorial contraction as listed in Tables 2.2 or 2.3. Then $K_{X_{i-1}} = K_{X_i} + \lambda \Gamma$ for some extremal curve Γ, and if $B_{i-1} \subset X_{i-1}$ is a curve with birational transform $B_i \subset X_i$, by projection formula we have

$$K_{X_i} \cdot B_i = \psi_i^* K_X \cdot B_{i-1} = (K_{X_{i-1}} - \lambda \Gamma) \cdot B_{i-1} < 0 \tag{2.14}$$

and $K_{X_{i-1}}^2 = K_{X_i}^2 + \lambda^2 \Gamma^2 < K_{X_i}$. Hence $-K_{X_i}$ is ample and X_i is a del Pezzo surface with singularities of type 2.12 as above.
Therefore, after a finite number of extremal contractions we obtain

$$X = X_0 \xrightarrow{\psi_1} X_1 \xrightarrow{\psi_2} \dots \xrightarrow{\psi k-1} X_{k-1} \xrightarrow{\psi_k} X_k \tag{2.15}$$

where either $\psi_k : X_{k-1} \to X_k$ is a fibration over $\mathbb{P}^1$ or $X_k = X_{\min}$ is a surface with $\rho(X_{\min}) = 1$ and singular locus of type 2.12; for the latter, by Lemma 2.2.1, there are only few possibilities for the types of singularities that a minimal del Pezzo surface with $\rho = 1$ can admit.
Moreover, if $\operatorname{Sing}(X_{\min}) = A_1 + 2 \times A_3$, then $X_{\min}$ is obtained by either contracting a surface of type $(2, 3)$ by $(C5) + 2 \times (C12)$ or of type $(3, 2)$ by $(C6) + 2 \times (C12)$. In both cases, the surfaces $X_{(2,3)}$ and $X_{(3,2)}$ would have $\rho = 4$, but that is not possible for such surfaces, as discussed in Section 1.5. Similarly, we can discard cases $2 \times A_1 + 2 \times A_3$, $4 \times A_2$ and $A_1 + 2 \times A_3 + \frac{1}{5}(1, 2)$.
If $\operatorname{Sing}(X_{\min}) = \{2 \times A_4\}$, then X comes from a surface of type $(0, 4)$ with $\rho = 3$, which does not appear as a numerical candidate.
If $\operatorname{Sing}(X_{\min}) = \{A_1 + 2 \times A_3\}$, then $X_{\min}$ comes from contractions of a surface X of type $(3, 2)$ respectively, but such surface would have $h^0(-K_X) = 0$.
All the remaining cases give surfaces which are listed in Table 2.4. Aside from the (weighted) projective spaces, we have:

- $\mathbb{P}^2/\mu_3$ is the quotient of $\mathbb{P}^2$ by the action of μ_3 with weights $(1, \zeta, \zeta^2)$. Its resolution is given by blowing up dP_6 in 3 intersection points of the (-1)–curves.

- $M_{2,2}$ is the toric surface represented by a complete intersection of two quadrics in $\mathbb{P}^4$.

- N_8 is a nontoric hypersurface defined by a polynomial of degree 8 in $\mathbb{P}(1, 1, 2, 5)$.

- $X^{5,3}_{(0,2)}$ is one of the toric surfaces listed in the tables in Section 5.2.2. The table shows vertices of the Fano polygon and invariants. We will discuss these surfaces in more detail in Chapter 3.

□

Remark 2.3.1. We know from the case $k \times \frac{1}{3}(1, 1)$ in [CH15] that the sequence of contractions $(\mathcal{C}11) + (\mathcal{C}2)$ is not minimal as other contractions with higher priority are available, so in our trees of possibilities it will not come up.

We now use the list of contractions to write down the possible outcomes of the MMP for the specified singularities. Indeed, by then applying some backwards engineering, we can reconstruct a (possibly nontoric) model for the resolution of our orbifold del Pezzo surfaces. We can recover the curve configurations for the minimal model by following the list of blow-ups from minimal surfaces of Picard rank 1 or from fibrations as listed in 2.3.1, so that we can exclude the cases for which, following the order stated in proposition 2.3.1, a different contraction could have been applied.
From now on, we will denote by (k_1, k_2) the case of surfaces with R-content $k_1 \times \frac{1}{3}(1, 1) + k_2 \times \frac{1}{5}(1, 2)$.
For the cases $(k_1, 0)$ we refer to [CH15], in particular in the cited Theorems 1.4.1, 1.4.2 and 1.4.3.
Recall from 1.1 we have a finite number of cases depending on the invariants.

Graphs for case analysis In the next sections we will analyse the possible contractions starting from a minimal surface with singularity type (k_1, k_2).
The graphs represent trees of possibilities indicating the possible contractions that a numerical candidate with specified singularity type can admit. Every horizontal branch will represent a purposed MMP for the surface, and all of the branches will be listed in numerical order. Every node describing a different endpoint of the same branch will be denoted by a letter.
Notice that in more than one case, at the said nodes we have more possibilities for a specific singularity type (e.g., if $X_{\min}$ is smooth, then we could have $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$).

We will discuss in detail cases $(0, 1)$ and $(1, 1)$. For what concerns the other cases, we report here the successful end results of the MMPs and we refer to the Appendix for more details about the calculations.

2.4 Case Analysis for $(k_1, 1)$

Tree for $(0, 1)$ For the case of orbifolds containing a singularity with R-content $\frac{1}{5}(1, 2)$ only, we have that the invariants K^2 and $h^0(-K)$ depend only on the Picard rank which ranges between 1 and 8.

$$\begin{tikzcd}
\rho \leq 8 & \rho \leq 7 & \rho \leq 6 \\
\frac{1}{5}(1,2) \arrow[r, "(\mathcal{C}5)"] \arrow[dr, "(\mathcal{C}8)"] & A_1 \arrow[r, "(\mathcal{C}1)"] & \text{smooth} \\
& A_2 \arrow[r, "(\mathcal{C}2)"] & \text{smooth}
\end{tikzcd}$$

Now, we will discuss the cases arising from the contractions leading to the nodes of the graph that admit a model surface with $\rho = 1$.

Case 1 (A) $(\mathcal{C}5) + (\mathcal{C}1)$: $\frac{1}{5} \longrightarrow$ smooth $\qquad \mathbb{P}^2$

By tracing back the subsequent blow ups in the minimal resolution, we see that in the resulting configuration of curves there is a (-1)-curve not intersecting any exceptional curve coming from the minimal resolution. This curve represents a floating (-1)-curve, so this surface is not minimal.

(B) $(\mathcal{C}5) + (\mathcal{C}1)$: $\frac{1}{5} \longrightarrow$ smooth $\qquad \mathbb{P}^1 \times \mathbb{P}^1$

Let us consider the configuration of extremal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ and the subsequent blow ups leading to the surface S.

The surface $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$ is a Hirzebruch surface, so in particular a fibration over $\mathbb{P}^1$ with rational fibres. Let $B \cong \mathbb{P}^1$ denote the base curve and $A \cong \mathbb{P}^1$ the fibre. These curves intersect transversely in one point and their self intersection is $A^2 = B^2 = 0$.

It is known that for a Hirzebruch surface $\mathbb{F}_a$ then $-K_{\mathbb{F}} = (a + 2)A + 2B$. If D is any curve in $\mathbb{F}_a$, then by adjunction formula we have

$$D^2 = -2 + [(a + 2)A + 2B] \cdot D$$

Thus, consider a curve $D \subset \mathbb{F}_0$ such that $D \cdot A = D \cdot B = 1$, then $D^2 = 2$. To obtain the surface S, consider first the blow ups corresponding to $(\mathcal{C}1)$ on the minimal resolution: they correspond to blowing up two infinitely near points, say p_1, p_2, where p_1 is a smooth point. As the curves A, B are base point free, then we consider the smooth point p_1 to be at the intersection of the two curves. The exceptional divisor E_1 coming from this blow up is such that $(E_1)^2 = -1$ and will intersect transversely the birational transforms A', B' of respectively A, B. In particular $A' = A - E_1$ and $B' = B - E_1$, so that $(A')^2 = (B')^2 = -1$. The curve D is affected by the blow up as well, since we assumed $p_1 \in D$. Thus $(D')^2 = 1$ and $D' \cdot A' = D' \cdot B' = 0$. Since we have taken the point p_2 to be infinitely near to p_1, then $p_2 \in E_1$, and blowing up such point gives another exceptional divisor E_2 such that $(E_2)^2 = -1$ and intersecting E_1 only. Specifically, if E_1' is the birational transform of E_1, then $E_1' = E_1 - E_2$, so $(E_1')^2 = -2$. This (-2)–curve in the minimal resolution identifies the A_1 singularity coming from reversing the divisorial contraction $(\mathcal{C}1)$. After this operation D' has not been affected by the blow up, thus we still have $(D')^2 = 1$.

Now, to obtain the surface S we need to blow up two more infinitely near points, say $p_3, p_4 \in E_1'$. By doing these operations, we will obtain a curve configuration with a (-3)–curve (i.e. the birational transform of E_1') and a (-2)–curve corresponding to the birational transform of the exceptional divisor appearing after the blow up of p_3. If we choose p_3 to be the point of intersection of D' with E_1', the two subsequent blow ups of the points p_3, p_4 will affect D': after the two operations we will have that D''', the birational transform of D', does not intersect any other exceptional curve but E_4 (where E_4 denotes the exceptional curve coming from the blow up of p_4).

In particular $(D''')^2 = -1$, and D''' is disjoint from the exceptional locus of the minimal resolution. This implied that D''' is a floating (-1)–curve and the surface obtained by this sequence of blow ups is not minimal. Therefore, this case does not give a good candidate.

(C) $(\mathcal{C}5):\ \frac{1}{5} \longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

For this case, we blow up a smooth point in the resolution of $\mathbb{P}(1,1,2)$, i.e. the Hirzebruch surface $\mathbb{F}_2$. Similarly as above, $A \cong \mathbb{P}^1$ is the fibre with $A^2 = 0$ and B is the rational curve with $B^2 = -2$. On $\mathbb{F}_2$ we have another extremal ray C such that $C = 2A + B$, so $C^2 = 2$. Thus, blowing up a smooth point of $\mathbb{F}_2$ means blowing up the intersection point of C with A.

Then, we obtain a nontoric configuration with invariants:

$$K^2 = \frac{32}{5} \qquad h^0(-K) = 7 \qquad \rho = 2 \qquad n = 3$$

This surface is minimal and we denote it by $S^3_{(0,1)}$.

Case 2 (A) $(\mathcal{C}8) + (\mathcal{C}2)$: $\frac{1}{5} \longrightarrow$ smooth $\qquad \mathbb{P}^2$

Again in this case we obtain a non-toric configuration with

$$K^2 = \frac{27}{5} \qquad h^0(-K) = 6 \qquad \rho = 3 \qquad n = 4$$

which is minimal and denoted by $S^4_{(0,1)}$.

(B) $(\mathcal{C}8) + (\mathcal{C}2)$: $\frac{1}{5} \longrightarrow$ smooth $\qquad \mathbb{P}^1 \times \mathbb{P}^1$

For the last case, we obtain two possibilities, both nontoric with one equivalent to the case (1), which is not minimal: indeed, in the minimal surface $\mathbb{F}_0$ leading to Y, take a curve D such that $D \cdot A = D \cdot B = 1$, where A, B are as defined in case (1B). Then reversing the two divisorial contractions $(\mathcal{C}8), (\mathcal{C}2)$ means blowing up two infinitely near point p_1, p_2 where p_1 is a smooth point on $\mathbb{F}_0$, and then two distinct points on the exceptional curve E_2 appearing after the blow up of the point p_2. After these blow ups, the birational transform of D will be a (-1)-curve that meets E_4 (i.e. the exceptional divisor over p_4) but it does not meet the exceptional locus of the resolution of singularities. So the birational transform of D represents a floating (-1)–curve making the resulting variety not minimal.

So, to conclude, we have 2 minimal surfaces with R-content $\frac{1}{5}(1,2)$:

(I) $S^3_{(0,1)}$ with $\rho = 2$ and $K^2 = \frac{32}{5}$

(II) $S^4_{(0,1)}$ with $\rho = 3$ and $K^2 = \frac{27}{5}$

Tree for $(1,1)$

$\rho \leq 7$		$\rho \leq 6$		$\rho \leq 5$		$\rho \leq 4$
$\frac{1}{3}+\frac{1}{5}$	$\xrightarrow{(\mathcal{C}5)}$	$A_1+\frac{1}{3}$	$\xrightarrow{(\mathcal{C}1)}$	$\frac{1}{3}$		
			$\xrightarrow{(\mathcal{C}6)}$	$2\times A_1$	$\xrightarrow{(\mathcal{C}1)}$	A_1
			$\xrightarrow{(\mathcal{C}7)}$	smooth		
	$\xrightarrow{(\mathcal{C}6)}$	$A_1+\frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$\frac{1}{5}$		
			$\xrightarrow{(\mathcal{C}8)}$	A_1+A_2	$\xrightarrow{(\mathcal{C}2)}$	A_1
			$\xrightarrow{(\mathcal{C}9)}$	smooth		
	$\xrightarrow{(\mathcal{C}8)}$	$A_2+\frac{1}{3}$	$\xrightarrow{(\mathcal{C}2)}$	$\frac{1}{3}$		
	$\xrightarrow{(\mathcal{C}10)}$	smooth				
	$\xrightarrow{(\mathcal{C}12)}$	A_3	$\xrightarrow{(\mathcal{C}3)}$	smooth		

The resulting cases are the following.

Case 1 $(\mathcal{C}5)+(\mathcal{C}1):\ \frac{1}{3}+\frac{1}{5}\longrightarrow\frac{1}{3} \qquad S_{(1,0)}$

After the two divisorial contractions $(\mathcal{C}5)+(\mathcal{C}1)$ we end up with a surface of type $(1,0)$ that is minimal. From [CH15] we have 2 cases of minimal surfaces with $\rho\leq 5$, namely $\mathbb{P}(1,1,3)$ with $\rho=1$ and $X_4\subset\mathbb{P}(1,1,1,3)$ with $\rho=4$;

(A) In the first case, by blowing up the minimal surface $\mathbb{P}(1,1,3)$ we obtain a nontoric surface $S^{3,1}_{(1,1)}$ with the following invariants:

$$K^2=\frac{71}{15} \qquad h^0(-K)=5 \qquad \rho=3 \qquad n=3$$

(B) To follow back the MMP from the surface $X_4\subset\mathbb{P}(1,1,1,3)$, take the configuration of curves of its minimal resolution and blow up two infinitely near points p_1,p_2 first, where p_1 is a smooth point, and then two more infinitely near points on the exceptional curve E_2. We obtain a non–minimal surface as the blow ups leading to the $\frac{1}{5}(1,2)$ singulatiy are locally isomorphic to the non minimal configuration of the case (1B) of the $(0,1)$ case analysis. Thus, by using the same choice of curve D we find a floating (-1)–curve.

Case 2 $(\mathcal{C}5) + (\mathcal{C}6) + (\mathcal{C}1) : \ \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 \qquad (\mathbb{P}(1,1,2))$

The blow ups induced on the minimal resolution of the minimal surface $\mathbb{P}(1,1,2)$ give a floating (-1)-curve, making S a non minimal surface.

Case 3 $(\mathcal{C}5) + (\mathcal{C}7) : \ \frac{1}{3} + \frac{1}{5} : \longrightarrow \text{smooth} \qquad (\mathbb{P}^2 \text{ or } \mathbb{P}^1 \times \mathbb{P}^1)$

In both cases, the blow ups give configurations where some contractions with higher priority are available, giving a non directed MMP.

Case 4 Similarly to case (1), we look at minimal surfaces with one $\frac{1}{5}(1,2)$ point, so from the analysis of case (0,1) we have 2 possible surfaces of this type with $\rho \leq 5$. For the first two cases we check blow ups from minimal surfaces $S^3_{0,1}$ and $S^4_{0,1}$, while we have a third case arising from the blow up of the minimal surface containing points $A_1 + \frac{1}{5}(1,2)$, i.e. $\mathbb{P}(1,2,5)$;

(A) $(\mathcal{C}6) + (\mathcal{C}1) : \ \frac{1}{3} + \frac{1}{5} \longrightarrow \frac{1}{5} \qquad (S^3_{0,1})$

The surface $S^4_{(1,1)}$ comes from blow ups of the minimal surface $S^3_{(0,1)}$ and admits the following invariants:

$$K^2 = \frac{56}{15} \qquad h^0(-K) = 2 \qquad \rho = 4 \qquad n = 4$$

(B) $(\mathcal{C}6) + (\mathcal{C}1) : \ \frac{1}{3} + \frac{1}{5} \longrightarrow \frac{1}{5} \qquad (S^4_{(0,1)})$

The surface coming from blow ups of the minimal surface $S^4_{(0,1)}$ has a floating (-1)-curve, making it non minimal.

(C) $(\mathcal{C}6) : \ \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 + \frac{1}{5} \qquad (\mathbb{P}(1,2,5))$

The configuration of curves arising from the blow ups of the minimal surface $\mathbb{P}(1,2,5)$ gives a toric surface $S^2_{(1,1)}$ with the following invariants:

$$K^2 = \frac{86}{15} \qquad h^0(-K) = 6 \qquad \rho = 2 \qquad n = 2$$

Case 5 (A) $(\mathcal{C}6) + (\mathcal{C}8) + (\mathcal{C}2) : \ \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 \qquad (\mathbb{P}(1,1,2))$

The surface resulting from the blow ups has a floating (-1)-curve, thus is not minimal.

(B) $(\mathcal{C}6) + (\mathcal{C}8) : \ \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 + A_2 \qquad (\mathbb{P}(1,2,3))$

The surface $S^{3,2}_{(1,1)}$ obtained by blowing up $\mathbb{P}(1,2,3)$ gives a non toric surface with following invariants:

$$K^2 = \frac{71}{15} \qquad h^0(-K) = 5 \qquad \rho = 3 \qquad n = 3$$

Case 6 $(\mathcal{C}6) + (\mathcal{C}9)$: $\frac{1}{3} + \frac{1}{5} \longrightarrow$ smooth $\qquad (\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1)$

In both cases the MMP ends up with give a surface where the MMP is not directed and other contractions with higher priority are available.

Case 7 $(\mathcal{C}8) + (\mathcal{C}2)$: $\frac{1}{3} + \frac{1}{5} \longrightarrow \frac{1}{3} \qquad S_{(1,0)}$

Similarly to case (1), the only surface of type $(0,1)$ available is $\mathbb{P}(1,1,3)$. The resulting surface has a non toric configuration $S^{3,3}_{(1,1)}$ with following invariants:

$$K^2 = \frac{71}{15} \qquad h^0(-K) = 5 \qquad \rho(X) = 3 \qquad n = 3$$

Case 8 $(\mathcal{C}10)$: $\frac{1}{3} + \frac{1}{5} \longrightarrow$ smooth $\qquad (\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1)$

In both cases the surfaces obtained from the blow ups are not minimal as contractions of type $(\mathcal{C}5)$ or $(\mathcal{C}6)$ are available.

Case 9 $(\mathcal{C}12) + (\mathcal{C}3)$: $\frac{1}{3} + \frac{1}{5} \longrightarrow$ smooth $\qquad (\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1)$

As above, in both smooth cases the blow ups of the minimal surfaces give non directed MMPs.

As a result, the minimal surfaces with $\frac{1}{3}(1,1) + \frac{1}{5}(1,2)$ points are

(I) $S^2_{(1,1)}$ with $\rho = 2$ and $K^2 = \frac{86}{15}$

(II) $S^{3,1}_{(1,1)}, S^{3,2}_{(1,1)}$ and $S^{3,3}_{(1,1)}$ with $\rho = 3$ and $K^2 = \frac{71}{15}$

(III) $S^4_{(1,1)}$ with $\rho = 4$ and $K^2 = \frac{56}{15}$

Tree for $(2,1)$

$\rho \leq 6$		$\rho \leq 5$		$\rho \leq 4$		$\rho \leq 3$		$\rho \leq 2$		$\rho \leq 1$
$2 \times \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}5)}$	$A_1 + 2 \times \frac{1}{3}$	$\xrightarrow{(\mathcal{C}1)}$	$2 \times \frac{1}{3}$						
			$\xrightarrow{(\mathcal{C}6)}$	$2 \times A_1 + \frac{1}{3}$	$\xrightarrow{(\mathcal{C}1)}$	$A_1 + \frac{1}{3}$	$\xrightarrow{(\mathcal{C}6)}$	$2 \times A_1$	$\xrightarrow{(\mathcal{C}1)}$	A_1
							$\xrightarrow{(\mathcal{C}7)}$	smooth		
					$\xrightarrow{(\mathcal{C}7)}$	A_1				
			$\xrightarrow{(\mathcal{C}7)}$	$\frac{1}{3}$						
			$\xrightarrow{(\mathcal{C}11)}$	$A_1 + A_3$	$\xrightarrow{(\mathcal{C}2)}$	A_1				
	$\xrightarrow{(\mathcal{C}6)}$	$A_1 + \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$\frac{1}{3} + \frac{1}{5}$						
			$\xrightarrow{(\mathcal{C}6)}$	$2 \times A_1 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$A_1 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}8)}$	$A_1 + A_2$	$\xrightarrow{(\mathcal{C}2)}$	A_1
							$\xrightarrow{(\mathcal{C}9)}$	smooth		
					$\xrightarrow{(\mathcal{C}9)}$	A_1				
			$\xrightarrow{(\mathcal{C}7)}$	$\frac{1}{5}$						
			$\xrightarrow{(\mathcal{C}9)}$	$\frac{1}{3}$						
			$\xrightarrow{(\mathcal{C}10)}$	A_1						
			$\xrightarrow{(\mathcal{C}12)}$	$A_1 + A_3$	$\xrightarrow{(\mathcal{C}3)}$	A_1				
	$\xrightarrow{(\mathcal{C}8)}$	$A_2 + 2 \times \frac{1}{3}$	$\xrightarrow{(\mathcal{C}2)}$	$2 \times \frac{1}{3}$						
	$\xrightarrow{(\mathcal{C}10)}$	$\frac{1}{3}$								
	$\xrightarrow{(\mathcal{C}11)}$	$A_2 + \frac{1}{5}$								
	$\xrightarrow{(\mathcal{C}12)}$	$A_3 + \frac{1}{3}$	$\xrightarrow{(\mathcal{C}3)}$	$\frac{1}{3}$						

Minimal surfaces:

$(\mathcal{C}6)+(\mathcal{C}1):\ 2\times\frac{1}{3}+\frac{1}{5}\longrightarrow\frac{1}{3}+\frac{1}{5}\qquad S^{3,3}_{(1,1)}$

$$S^{4}_{(2.1)}:\qquad K^2=\frac{31}{15}\qquad h^0(-K)=2\qquad \rho=5\qquad n=4$$

$(\mathcal{C}10):\ 2\times\frac{1}{3}+\frac{1}{5}\longrightarrow\frac{1}{3}\qquad \mathbb{P}(1,1,3)$

$$S^{1,1}_{(2,1)}:\qquad K^2=\frac{76}{15}\qquad h^0(-K)=5\qquad \rho=2\qquad n=1$$

$(\mathcal{C}11):\ 2\times\frac{1}{3}+\frac{1}{5}\longrightarrow A_2+\frac{1}{5}\qquad \mathbb{P}(1,3,5)$

$$S^{1,2}_{(2,1)}:\qquad K^2=\frac{76}{15}\qquad h^0(-K)=5\qquad \rho=2\qquad n=1$$

Tree for $(3,1)$

$\rho \leq 5$		$\rho \leq 4$		$\rho \leq 3$		$\rho \leq 2$		$\rho \leq 1$
$3 \times \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}5)}$	$A_1 + 3 \times \frac{1}{3}$	$\xrightarrow{(\mathcal{C}1)}$	$3 \times \frac{1}{3}$				
			$\xrightarrow{(\mathcal{C}6)}$	$2 \times A_1 + 2 \times \frac{1}{3}$	$\xrightarrow{(\mathcal{C}1)}$	$A_1 + 2 \times \frac{1}{3}$	$\xrightarrow{(\mathcal{C}7)}$	$\frac{1}{3}$
							$\xrightarrow{(\mathcal{C}11)}$	$A_1 + A_2$
					$\xrightarrow{(\mathcal{C}7)}$	$A_1 + \frac{1}{3}$	$\xrightarrow{(\mathcal{C}7)}$	$\mathbb{P}^2$
			$\xrightarrow{(\mathcal{C}7)}$	$2 \times \frac{1}{3}$				
	$\xrightarrow{(\mathcal{C}6)}$	$A_1 + 2 \times \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$2 \times \frac{1}{3} + \frac{1}{5}$				
			$\xrightarrow{(\mathcal{C}6)}$	$2 \times A_1 + \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$A_1 + \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}9)}$	$\frac{1}{3}$
							$\xrightarrow{(\mathcal{C}10)}$	A_1
					$\xrightarrow{(\mathcal{C}7)}$	$A_1 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}8)}$	$A_1 + A_2$
							$\xrightarrow{(\mathcal{C}9)}$	smooth
					$\xrightarrow{(\mathcal{C}8)}$	$2 \times A_1 + A_2 + \frac{1}{3}$ $(\mathcal{F}1 + \mathcal{F}2)$		
					$\xrightarrow{(\mathcal{C}12)}$	$2 \times A_1 + A_3$ $(\mathcal{F}0 + \mathcal{F}1)$		
			$\xrightarrow{(\mathcal{C}7)}$	$\frac{1}{3} + \frac{1}{5}$				
			$\xrightarrow{(\mathcal{C}8)}$	$A_1 + A_2 + 2 \times \frac{1}{3}$	$\xrightarrow{(\mathcal{C}2)}$	$A_1 + 2 \times \frac{1}{3}$	$\xrightarrow{(\mathcal{C}9)}$	$\frac{1}{3}$
	$\xrightarrow{(\mathcal{C}8)}$	$A_2 + 3 \times \frac{1}{3}$	$\xrightarrow{(\mathcal{C}2)}$	$3 \times \frac{1}{3}$				
	$\xrightarrow{(\mathcal{C}10)}$	$2 \times \frac{1}{3}$						
	$\xrightarrow{(\mathcal{C}12)}$	$A_3 + 2 \times \frac{1}{3}$	$\xrightarrow{(\mathcal{C}3)}$	$\frac{1}{3} + \frac{1}{5}$				

$(\mathcal{C}6) + (\mathcal{C}1): \; 3 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \qquad S_{(2,1)}$

$$S^2_{(3,1)}: \qquad K^2 = \frac{12}{5} \qquad h^0(-K) = 2 \qquad \rho = 4 \qquad n = 2$$

Tree for $(4,1)$

$\rho \leq 4$	$\rho \leq 3$	$\rho \leq 2$	$\rho \leq 1$
$4 \times \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}6)}$ $A_1 + 3 \times \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}7)}$ $2 \times \frac{1}{3} + \frac{1}{5}$	
		$\xrightarrow{(\mathcal{C}10)}$ $A_1 + 2 \times \frac{1}{3}$	$\xrightarrow{(\mathcal{C}11)}$ $A_1 + A_2$
	$\xrightarrow{(\mathcal{C}8)}$ $A_2 + 4 \times \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}11)}$ $2 \times A_2 + 2 \times \frac{1}{3}$	$\xrightarrow{(\mathcal{C}11)}$ $3 \times A_2$

$(\mathcal{C}6) + (\mathcal{C}7):\ \ 4 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \qquad S^{1,1}_{(2,1)}$

$$S^1_{(4,1)}: \qquad K^2 = \frac{47}{15} \qquad h^0(-K) = 1 \qquad \rho = 4 \qquad n = 1$$

2.5 Case Analysis for $(k_1, 2)$

Tree for $(0, 2)$

$\rho \leq 6$		$\rho \leq 5$		$\rho \leq 4$		$\rho \leq 3$
$2 \times \frac{1}{5}$	$\xrightarrow{(\mathcal{C}5)}$	$A_1 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$\frac{1}{5}$		
			$\xrightarrow{(\mathcal{C}5)}$	$2 \times A_1$	$\xrightarrow{(\mathcal{C}1)}$	A_1
			$\xrightarrow{(\mathcal{C}8)}$	$A_1 + A_2$	$\xrightarrow{(\mathcal{C}2)}$	A_1
			$\xrightarrow{(\mathcal{C}9)}$	smooth		
	$\xrightarrow{(\mathcal{C}8)}$	$A_2 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}2)}$	$\frac{1}{5}$		
	$\xrightarrow{(\mathcal{C}13)}$	A_4	$\xrightarrow{(\mathcal{C}4)}$	smooth		

$$(\mathcal{C}5): \ 2 \times \tfrac{1}{5}(1,2) \longrightarrow A_1 + \tfrac{1}{5}(1,2) \qquad \mathbb{P}(1,2,5)$$

$$S^{2,1}_{(0,2)}: \qquad K^2 = \frac{24}{5} \qquad h^0(-K) = 5 \qquad \rho = 2 \qquad n = 2$$

$$(\mathcal{C}8): \ 2 \times \tfrac{1}{5}(1,2) \longrightarrow A_2 + \tfrac{1}{5}(1,2) \qquad \mathbb{P}(1,3,5)$$

$$S^{2,2}_{(0,2)}: \qquad K^2 = \frac{24}{5} \qquad h^0(-K) = 5 \qquad \rho = 2 \qquad n = 2$$

$$(\mathcal{C}8) + (\mathcal{C}2): \ 2 \times \tfrac{1}{5}(1,2) \longrightarrow \tfrac{1}{5}(1,2) \qquad S^4_{(0,1)}$$

$$S^5_{(0,2)}: \qquad K^2 = \frac{9}{5} \qquad h^0(-K) = 2 \qquad \rho = 5 \qquad n = 5$$

Tree for $(1,2)$

$\rho \leq 5$		$\rho \leq 4$		$\rho \leq 3$		$\rho \leq 2$		$\rho \leq 1$
$\frac{1}{3} + 2 \times \frac{1}{5}$	$\xrightarrow{(\mathcal{C}5)}$	$A_1 + \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$\frac{1}{3} + \frac{1}{5}$				
			$\xrightarrow{(\mathcal{C}5)}$	$2 \times A_1 + \frac{1}{3}$	$\xrightarrow{(\mathcal{C}1)}$	$A_1 + \frac{1}{3}$	$\xrightarrow{(\mathcal{C}7)}$	smooth
					$\xrightarrow{(\mathcal{C}7)}$	A_1		
			$\xrightarrow{(\mathcal{C}6)}$	$2 \times A_1 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$A_1 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}8)}$	$A_1 + A_2$
							$\xrightarrow{(\mathcal{C}9)}$	smooth
					$\xrightarrow{(\mathcal{C}9)}$	A_1		
			$\xrightarrow{(\mathcal{C}7)}$	$\frac{1}{5}$				
			$\xrightarrow{(\mathcal{C}9)}$	$\frac{1}{3}$				
			$\xrightarrow{(\mathcal{C}10)}$	A_1				
			$\xrightarrow{(\mathcal{C}12)}$	$A_1 + A_2$	$\xrightarrow{(\mathcal{C}3)}$	A_1		
	$\xrightarrow{(\mathcal{C}6)}$	$A_1 + 2 \times \frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$2 \times \frac{1}{5}$				
			$\xrightarrow{(\mathcal{C}8)}$	$A_1 + A_2 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}2)}$	$A_1 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}8)}$	$A_1 + A_2$
							$\xrightarrow{(\mathcal{C}9)}$	smooth
			$\xrightarrow{(\mathcal{C}9)}$	$\frac{1}{5}$				
			$\xrightarrow{(\mathcal{C}13)}$	$A_1 + A_4$	$\xrightarrow{(\mathcal{C}4)}$	A_1		
	$\xrightarrow{(\mathcal{C}8)}$	$A_2 + \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}2)}$	$\frac{1}{3} + \frac{1}{5}$				
	$\xrightarrow{(\mathcal{C}13)}$	$A_4 + \frac{1}{3}$	$\xrightarrow{(\mathcal{C}4)}$	$\frac{1}{3}$				

$(\mathcal{C}5) + (\mathcal{C}5) + (\mathcal{C}7): \ \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

$$S^{3,1}_{(1,2)}: \qquad K^2 = \frac{32}{15} \qquad h^0(-K) = 2 \qquad \rho = 4 \qquad n = 3$$

$(\mathcal{C}5) + (\mathcal{C}6) + (\mathcal{C}1): \ \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow A_1 + \frac{1}{5}(1,2) \qquad \mathbb{P}(1,2,5)$

$$S^{3,2}_{(1,2)}: \qquad K^2 = \frac{32}{15} \qquad h^0(-K) = 2 \qquad \rho = 4 \qquad n = 3$$

$(\mathcal{C}5) + (\mathcal{C}9): \ \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow \frac{1}{3}(1,1) \qquad \mathbb{P}(1,1,3)$

$$S^2_{(1,2)}: \qquad K^2 = \frac{62}{15} \qquad h^0(-K) = 3 \qquad \rho = 3 \qquad n = 2$$

$(\mathcal{C}6) + (\mathcal{C}1): \ \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{5}(1,2) \qquad S^{2,2}_{(0,2)}$

$$S^{3,3}_{1,2}: \qquad K^2 = \frac{32}{15} \qquad h^0(-K) = 2 \qquad \rho = 4 \qquad n = 3$$

Tree for $(2,2)$

$\rho \le 4$		$\rho \le 3$		$\rho \le 2$		$\rho \le 1$
$2 \times \frac{1}{3} + 2 \times \frac{1}{5}$	$\xrightarrow{(\mathcal{C}5)}$	$A_1 + 2 \times \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$2 \times \frac{1}{3} + \frac{1}{5}$		
			$\xrightarrow{(\mathcal{C}6)}$	$2 \times A_1 + \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}7)}$	$A_1 + \frac{1}{5}$
					$\xrightarrow{(\mathcal{C}12)}$	$2 \times A_1 + A_3$
			$\xrightarrow{(\mathcal{C}7)}$	$\frac{1}{3} + \frac{1}{5}$		
	$\xrightarrow{(\mathcal{C}6)}$	$A_1 + \frac{1}{3} + 2 \times \frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$\frac{1}{3} + 2 \times \frac{1}{5}$		
			$\xrightarrow{(\mathcal{C}6)}$	$2 \times A_1 + 2 \times \frac{1}{5}$ $(\mathcal{F}0 + \mathcal{F}3)$	$\xrightarrow{(\mathcal{C}9)}$	$A_1 + \frac{1}{5}$
			$\xrightarrow{(\mathcal{C}7)}$	$2 \times \frac{1}{5}$		
			$\xrightarrow{(\mathcal{C}8)}$	$A_1 + A_2 + \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}10)}$	$A_1 + A_2$
			$\xrightarrow{(\mathcal{C}9)}$	$\frac{1}{3} + \frac{1}{5}$		
			$\xrightarrow{(\mathcal{C}10)}$	$A_1 + \frac{1}{5}$		
			$\xrightarrow{(\mathcal{C}12)}$	$A_1 + A_3 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}3)}$	$A_1 + \frac{1}{5}$
			$\xrightarrow{(\mathcal{C}13)}$	$A_1 + A_4 + \frac{1}{5}$		
	$\xrightarrow{(\mathcal{C}8)}$	$A_2 + 2 \times \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}2)}$	$2 \times \frac{1}{3} + \frac{1}{5}$		
	$\xrightarrow{(\mathcal{C}10)}$	$\frac{1}{3} + \frac{1}{5}$				
	$\xrightarrow{(\mathcal{C}12)}$	$A_3 + \frac{1}{3} + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}12)}$	$2 \times A_3$ $(2 \times \mathcal{F}_0)$		

$$(\mathcal{C}5) + (\mathcal{C}1): \; 2 \times \tfrac{1}{3}(1,1) + 2 \times \tfrac{1}{5}(1,2) \longrightarrow 2 \times \tfrac{1}{3}(1,1) + \tfrac{1}{5}(1,2) \qquad S^{1,1}_{(2,1)}$$

$$S^{2,1}_{(2,2)}: \qquad K^2 = \frac{22}{15} \qquad h^0(-K) = 1 \qquad \rho = 4 \qquad n = 2$$

$$(\mathcal{C}5)+(\mathcal{C}1):\ \ 2\times\tfrac{1}{3}(1,1)+2\times\tfrac{1}{5}(1,2)\longrightarrow 2\times\tfrac{1}{3}(1,1)+\tfrac{1}{5}(1,2)\qquad S^{1,2}_{(2,1)}$$

$$S^{2,2}_{(2,2)}:\qquad K^2\frac{22}{15}\qquad h^0(-K)=1\qquad \rho=4\qquad n=2$$

$$(\mathcal{C}6)+(\mathcal{C}10):\ \ 2\times\tfrac{1}{3}(1,1)+2\times\tfrac{1}{5}(1,2)\longrightarrow A_1+\tfrac{1}{5}(1,2)\qquad \mathbb{P}(1,2,5)$$

$$S^{1}_{(2,2)}:\qquad K^2=\frac{37}{15}\qquad h^0(-K)=2\qquad \rho=3\qquad n=1$$

$$(\mathcal{C}8)+(\mathcal{C}2):\ \ 2\times\tfrac{1}{3}(1,1)+2\times\tfrac{1}{5}(1,2)\longrightarrow 2\times\tfrac{1}{3}(1,1)+\tfrac{1}{5}(1,2)\qquad S^{1,1}_{(2,1)}$$

$$S^{2,3}_{(2,2)}:\qquad K^2=\frac{22}{15}\qquad h^0(-K)=1\qquad \rho=4\qquad n=2$$

$$(\mathcal{C}8)+(\mathcal{C}2):\ \ 2\times\tfrac{1}{3}(1,1)+2\times\tfrac{1}{5}(1,2)\longrightarrow 2\times\tfrac{1}{3}(1,1)+\tfrac{1}{5}(1,2)\qquad S^{1,2}_{(2,1)}$$

$$S^{2,4}_{(2,2)}:\qquad K^2=\frac{22}{15}\qquad h^0(-K)=1\qquad \rho=4\qquad n=2$$

Tree for $(3,2)$

$$\begin{array}{ccccc}
\rho \leq 3 & & \rho \leq 2 & & \rho \leq 1 \\
3 \times \frac{1}{3} + 2 \times \frac{1}{5} & \xrightarrow{(\mathcal{C}10)} & 2 \times \frac{1}{3} + \frac{1}{5} & & \\
& \xrightarrow{(\mathcal{C}11)} & A_2 + \frac{1}{3} + 2 \times \frac{1}{5} & \xrightarrow{(\mathcal{C}12)} & A_2 + A_3 + \frac{1}{5} \\
& & (\mathcal{F}2 + \mathcal{F}3) & &
\end{array}$$

$(\mathcal{C}10):\ 3 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \qquad S^{1,1}_{(2,1)}$

$$S^{0,1}_{(3,2)}: \qquad K^2 = \frac{9}{5} \qquad h^0(-K) = 1 \qquad \rho = 3 \qquad n = 0$$

$(\mathcal{C}10):\ 3 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \qquad S^{1,2}_{(2,1)}$

$$S^{0,2}_{(3,2)}: \qquad K^2 = \frac{9}{5} \qquad h^0(-K) = 1 \qquad \rho = 3 \qquad n = 0$$

2.6 Case Analysis for $(k_1, 3)$

Tree for $(0, 3)$

$\rho \le 4$		$\rho \le 3$		$\rho \le 2$		$\rho \le 1$
$3 \times \frac{1}{5}$	$\xrightarrow{(\mathcal{C}5)}$	$A_1 + 2 \times \frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$2 \times \frac{1}{5}$		
			$\xrightarrow{(\mathcal{C}5)}$	$2 \times A_1 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}1)}$	$A_1 + \frac{1}{5}$
					$\xrightarrow{(\mathcal{C}9)}$	A_1
			$\xrightarrow{(\mathcal{C}8)}$	$A_1 + A_2 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}2)}$	$A_1 + \frac{1}{5}$
			$\xrightarrow{(\mathcal{C}9)}$	$\frac{1}{5}$		
			$\xrightarrow{(\mathcal{C}13)}$	$A_1 + A_4$	$\xrightarrow{(\mathcal{C}4)}$	A_1
	$\xrightarrow{(\mathcal{C}8)}$	$A_2 + 2 \times \frac{1}{5}$	$\xrightarrow{(\mathcal{C}2)}$	$2 \times \frac{1}{5}$		
			$\xrightarrow{(\mathcal{C}8)}$	$2 \times A_2 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}2)}$	$A_2 + \frac{1}{5}$
	$\xrightarrow{(\mathcal{C}13)}$	$A_4 + \frac{1}{5}$	$\xrightarrow{(\mathcal{C}4)}$	$\frac{1}{5}$		

$(\mathcal{C}5) + (\mathcal{C}1):\ 3 \times \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{5}(1,2) \qquad S^{2,1}_{(0,2)}$

$$S^{3,1}_{(0,3)}: \qquad K^2 = \frac{6}{5} \qquad h^0(-K) = 1 \qquad \rho = 4 \qquad n = 3$$

$(\mathcal{C}5) + (\mathcal{C}1):\ 3 \times \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{5}(1,2) \qquad S^{2,2}_{(0,2)}$

$$S^{3,2}_{(0,3)}: \qquad K^2 = \frac{6}{5} \qquad h^0(-K) = 1 \qquad \rho = 4 \qquad n = 3$$

Tree for $(1,3)$

$\rho \leq 3$	$\rho \leq 2$	$\rho \leq 1$
$\frac{1}{3} + 3 \times \frac{1}{5}$	$\xrightarrow{(\mathcal{C}5)}$ $A_1 + \frac{1}{3} + 2 \times \frac{1}{5}$	$\xrightarrow{(\mathcal{C}10)}$ $A_1 + \frac{1}{5}$
		$\xrightarrow{(\mathcal{C}13)}$ $A_1 + A_4 + \frac{1}{5}$
	$\xrightarrow{(\mathcal{C}8)}$ $A_2 + \frac{1}{3} + 2 \times \frac{1}{5}$ $(\mathcal{F}2 + \mathcal{F}3)$	$\xrightarrow{(\mathcal{C}10)}$ $A_2 + \frac{1}{5}$
		$\xrightarrow{(\mathcal{C}12)}$ $A_2 + A_3 + \frac{1}{5}$
	$\xrightarrow{(\mathcal{C}10)}$ $2 \times \frac{1}{5}$	
	$\xrightarrow{(\mathcal{C}12)}$ $A_3 + 2 \times \frac{1}{5}$ $(\mathcal{F}0 + \mathcal{F}3)$	

$(\mathcal{C}5) + (\mathcal{C}10): \ \frac{1}{3}(1,1) + 3 \times \frac{1}{5}(1,2) \longrightarrow A_1 + \frac{1}{5}(1,2) \qquad \mathbb{P}(1,2,5)$

$$S^{1,1}_{(1,3)}: \qquad K^2 = \frac{23}{15} \qquad h^0(-K) = 1 \qquad \rho = 3 \qquad n = 1$$

$(\mathcal{C}8) + (\mathcal{C}10): \ \frac{1}{3}(1,1) + 3 \times \frac{1}{5}(1,2) \longrightarrow A_1 + \frac{1}{5}(1,2) \qquad \mathbb{P}(1,3,5)$

$$S^{1,2}_{(1,3)}: \qquad K^2 = \frac{25}{15} \qquad h^0(-K) = 1 \qquad \rho = 3 \qquad n = 1$$

$(\mathcal{C}8) + (\mathcal{C}12): \ \frac{1}{3}(1,1) + 3 \times \frac{1}{5}(1,2) \longrightarrow A_2 + A_3 + \frac{1}{5}(1,2) \qquad \mathbb{P}(3,4,5)$

$$S^{1,3}_{(1,3)}: \qquad K^2 = \frac{25}{15} \qquad h^0(-K) = 1 \qquad \rho = 3 \qquad n = 1$$

2.7 Case Analysis for $(k_1, 4)$

Tree for $(0, 4)$

$$\begin{array}{ccc} \rho \leq 2 & & \rho \leq 1 \\ & & \\ 4 \times \frac{1}{5} & \xrightarrow{(\mathcal{C}13)} & A_4 + 2 \times \frac{1}{5} \\ (2 \times \mathcal{F}3) & & \end{array}$$

$(\mathcal{C}13)$: $4 \times \frac{1}{5}(1,2) \longrightarrow R$

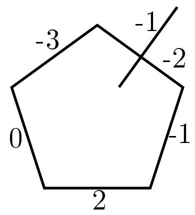
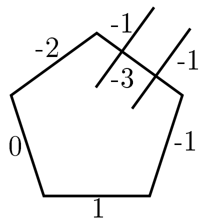
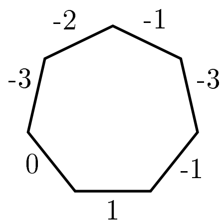
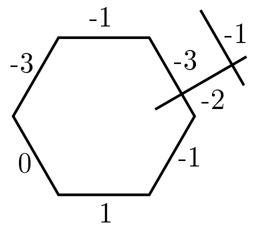
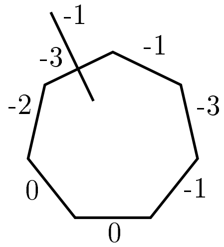
$$S^{0,1}_{(0,4)} : \qquad K^2 = \frac{8}{5} \qquad h^0(-K) = 1 \qquad \rho = 2 \qquad n = 0$$

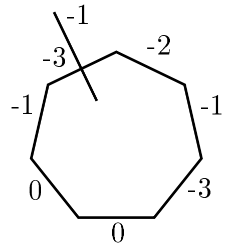
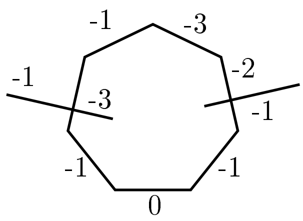
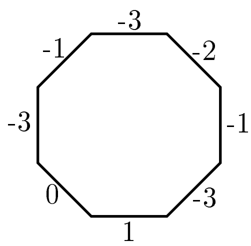
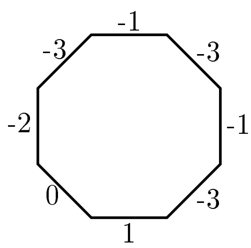
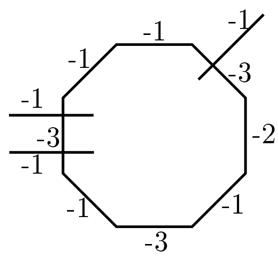
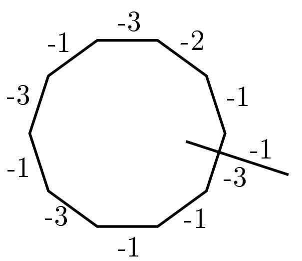
$4 \times \frac{1}{5}(1,2) \longrightarrow 2 \times (\mathcal{F}3)$

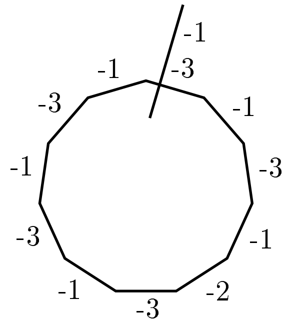
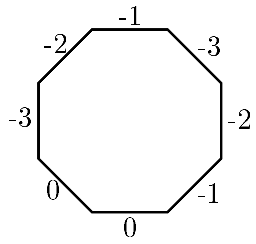
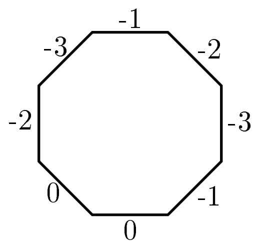
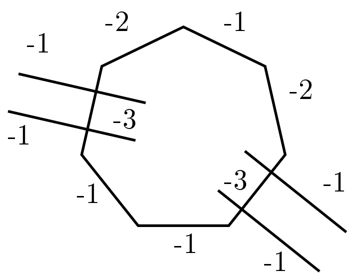
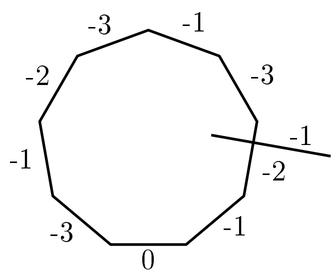
$$S^{0,2}_{(0,4)} : \qquad K^2 = \frac{8}{5} \qquad h^0(-K) = 1 \qquad \rho = 2 \qquad n = 0$$

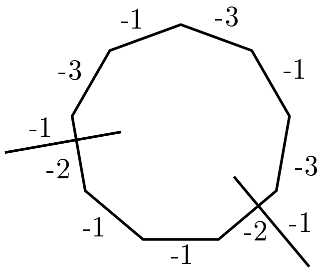
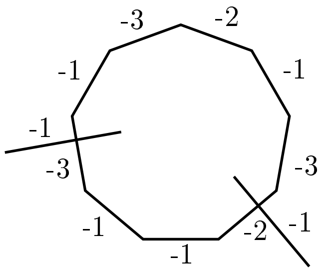
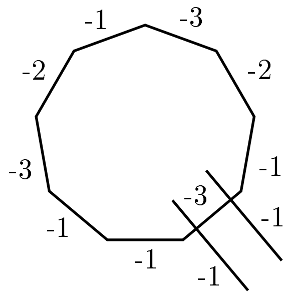
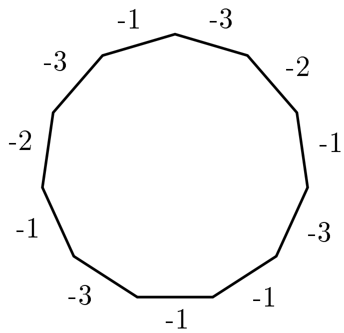
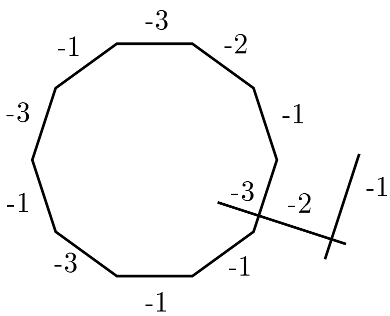
2.8 Minimal Surfaces

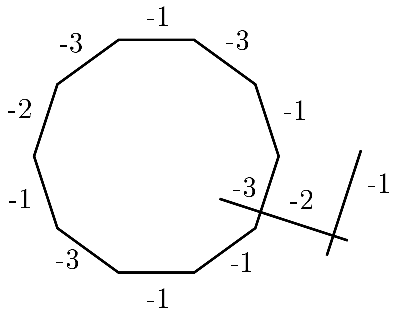
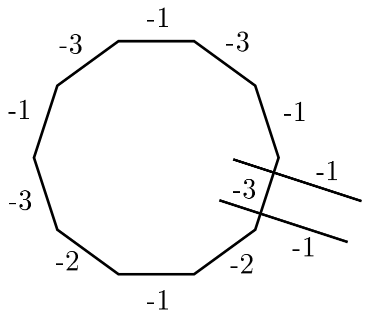
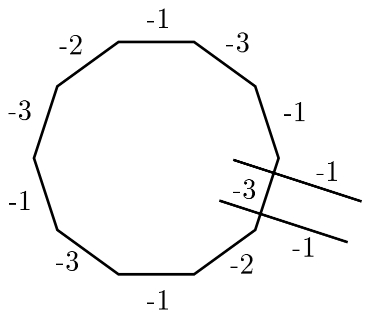
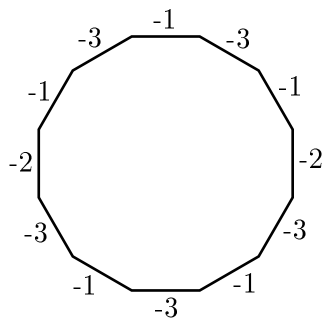
Theorem 2.8.1. *There are 33 isomorphism classes of minimal del Pezzo Orbifolds, $h^0(-K_X) \neq 0$ and admitting $\{k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2)\}$ singularities ($k_2 \geq 1$).*
The curve configurations of their resolutions are listed in Table 2.5 below.
Their birational models are recapped in Table 5.1 in Chapter 5.

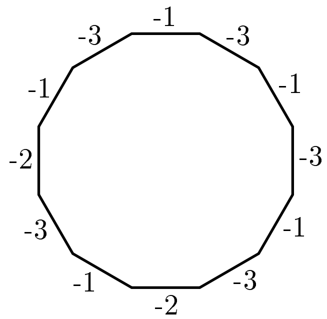
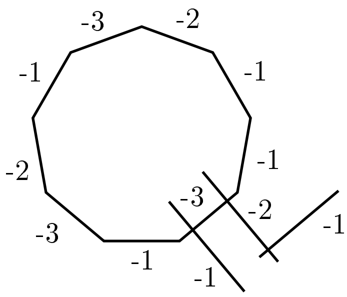
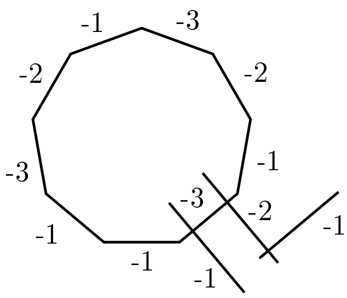
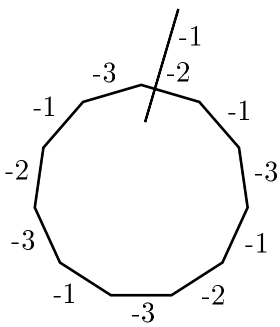
Surface	$\rho(X)$	K^2	Curve Configuration
$S^3_{(0,1)}$	2	$\frac{32}{5}$	
$S^4_{(0,1)}$	3	$\frac{27}{5}$	
$S^2_{(1,1)}$	2	$\frac{86}{15}$	
$S^{3,1}_{(1,1)}$	3	$\frac{71}{15}$	
$S^{3,2}_{(1,1)}$	3	$\frac{71}{15}$	

Surface	$\rho(X)$	K^2	Curve Configuration
$S^{3,3}_{(1,1)}$	3	$\frac{71}{15}$	
$S^{4}_{(1,1)}$	4	$\frac{56}{15}$	
$S^{1,1}_{(2,1)}$	2	$\frac{76}{15}$	
$S^{1,2}_{(2,1)}$	2	$\frac{76}{15}$	
$S^{4}_{(2,1)}$	5	$\frac{31}{15}$	
$S^{2}_{(3,1)}$	4	$\frac{12}{5}$	

Surface	$\rho(X)$	K^2	Curve Configuration
$S^1_{(4,1)}$	4	$\frac{26}{15}$	
$S^{2,1}_{(0,2)}$	2	$\frac{24}{5}$	
$S^{2,2}_{(0,2)}$	2	$\frac{24}{5}$	
$S^5_{(0,2)}$	5	$\frac{9}{5}$	
$S^2_{(1,2)}$	3	$\frac{47}{15}$	

Surface	$\rho(X)$	K^2	Curve Configuration
$S^{3,1}_{(1,2)}$	4	$\frac{32}{15}$	
$S^{3,2}_{(1,2)}$	4	$\frac{32}{15}$	
$S^{3,3}_{(1,2)}$	4	$\frac{32}{15}$	
$S^{1}_{(2,2)}$	3	$\frac{37}{15}$	
$S^{2,1}_{(2,2)}$	4	$\frac{22}{15}$	

Surface	$\rho(X)$	K^2	Curve Configuration
$S^{2,2}_{(2,2)}$	4	$\frac{22}{15}$	
$S^{2,3}_{(2,2)}$	4	$\frac{22}{15}$	
$S^{2,4}_{(2,2)}$	4	$\frac{22}{15}$	
$S^{0,1}_{(3,2)}$	3	$\frac{9}{5}$	

Surface	$\rho(X)$	K^2	Curve Configuration
$S^{0,2}_{(3,2)}$	3	$\frac{9}{5}$	
$S^{3,1}_{(0,3)}$	4	$\frac{6}{5}$	
$S^{3,2}_{(0,3)}$	4	$\frac{6}{5}$	
$S^{1,1}_{(1,3)}$	3	$\frac{23}{15}$	

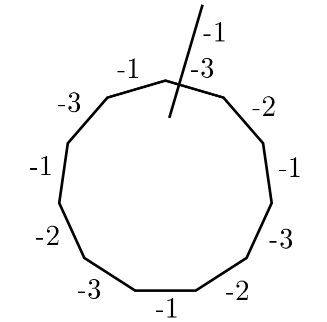
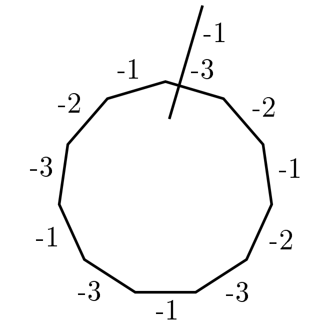
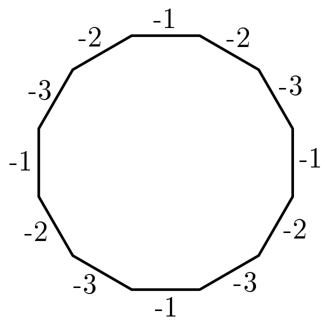
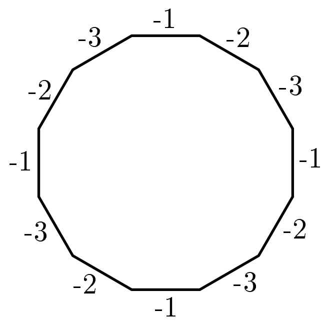
Surface	$\rho(X)$	K^2	Curve Configuration
$S^{1,2}_{(1,3)}$	3	$\frac{23}{15}$	
$S^{1,3}_{(1,3)}$	3	$\frac{23}{15}$	
$S^{0,1}_{(0,4)}$	2	$\frac{8}{5}$	
$S^{0,2}_{(0,4)}$	2	$\frac{8}{5}$	

Table 2.5: Minimal surfaces of type (k_1, k_2)

2.9 The case $h^0(-K_X) = 0$

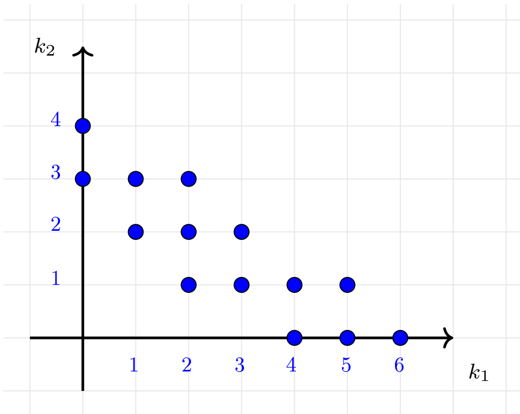
So far we have considered surfaces with $h^0(-K_X) \neq 0$, and we have found all of the possible minimal surfaces with said singularity types.
Let us now consider the case of minimal surfaces with $h^0(-K_X) = 0$. From the calculations carried out in Section 1.5, we have that for a surface of type (k_1, k_2) with $h^0(-K_X) = 0$ and Picard rank ρ, the following must hold:

$$\begin{gathered} 1 + K_X^2 - \tfrac{1}{3}k_1 - \tfrac{2}{5}k_2 = 0 \\ \rho = 11 - k_1 - 2k_2 \end{gathered} \tag{2.16}$$

Moreover, as $n \geq 0$ and $K_X^2 > 0$ then we have the following system of inequalities:

$$\begin{cases} 2k_1 + 3k_2 \leq 13 \\ 5k_1 + 6k_2 > 15 \end{cases} \tag{2.17}$$

The integer solutions to the system give the following numerical candidates for the said surfaces:



From the diagram, we can see that the possible singularity types are a subset of the pairs listed in 1.5, plus two more singularity types (namely $(5, 1)$ and $(2, 3)$) for surfaces that arise only in case $h^0(-K_X) = 0$. By construction, the latter two cases must be minimal. For the other singularity types, such surfaces can either appear as minimal or as blow ups of surfaces in the cascade for the given singularity content.

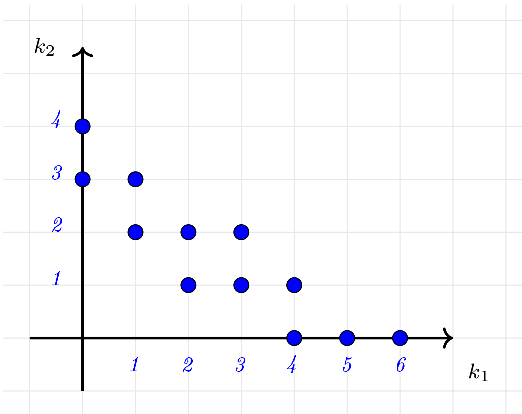
To check whether such surfaces exist in the minimal case, we turn again to the trees of possibilities: we can construct graphs as described above knowing that the Picard rank is fixed by 2.16. In particular, for the cases we have already analysed, it is sufficient to check whehter the given trees can be extended so that the initial surface has

$\rho = 11 - k_1 - 2k_2$. It turns out that no branch in this analysis gives a minimal surface, thus for the singularity types listed in 1.5, surfaces with $h^0(-K_X) = 0$ only arise as blow ups of other surfaces with same singularity type.
For the two cases $(5, 1)$ and $(2, 3)$, the surfaces will have $\rho = 4$ and $\rho = 3$ respectively. If we try to draw up a tree of possiblities, at the enpoints we get a surface with singular locus of type

$$\mathrm{Sing}(X) = \{k_1 \times \frac{1}{3}(1,1), k_2 \times \frac{1}{5}(1,2), n_1 \times A_1, n_2 \times A_2, n_3 \times A_3, n_4 \times A_4\}$$

but with either no minimal representative or wrong Picard rank. Therefore, there are no del Pezzo orbifolds with singularity type $(5, 1)$ or $(2, 3)$.
To conclude, we have the following:

Propostion 2.9.1. *Let X be a del Pezzo orbifold of type (k_1, k_2) and $h^0(-K_X) = 0$. Then the singularity type is one of the following*



and X can only appear as blow up of a (not necessarily minimal) surface with said singularity type.

Chapter 3

Toric Degenerations

This chapter will be devoted to the treatment of toric varieties and their deformations. In particular we will find the relations with the minimal surfaces listed in Chapter 3.

3.1 Classification of Polygons with specified singularity content

In Chapter 1 we have defined the notion of LDP-polygon, and described the relation with orbifold del Pezzo surfaces. Moreover, from 1.2.5 we know that if there exists a mutation between two Fano polygons P, P' then the respective toric surfaces are qG–equivalent. From [KNP15], there is a notion of *minimality* for mutation classes of polygons, such that a minimal polygon can serve as the representative of the qG–deformation class. Specifically, if ∂P denotes the boundary of the Fano polygon P, then we have:

Definition 3.1.1 ([KNP15], Definition 4.1)**.** A Fano polygon $P \subset N$ is called **minimal** if for every mutation equivalent polygon P' we have

$$\mid \partial P \cap N \mid \leq \mid \partial P' \cap N \mid$$

With the help of computer algebra we can find all the qG–classes of toric del Pezzo surfaces up to mutation and list all possible candidates for the toric degenerations. This is done by following several steps:

1. By using the algorithm described in [CK17], we can find all the possible minimal polygons with maximal index 5 (up to isomorphism) given the singularity content $(n, k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2))$;

2. Following the algorithm in [KNP15], we can make sure that there are no minimal

polygons of higher maximal local index, thus every minimal polygon appears with maximal local index 5;

3. Given the list of polygons, we will have to check that each of them represents a distinct mutation class; in order to do this, we use two criteria:

   - Polygons having finite mutation type: the algorithm described in [CK17] determines the mutation tree of length 8, so there might still be cases in the list that are mutation equivalent linked by a chain of 9 or more mutations. In order to check that this s not the case, ([Pri1]) gives a method to understand the number of mutations linking two polygons with same singularity content. Indeed, in the case the polygon P has one mutable edge, not only is the polygon of finite mutation type, but we can explicitly contruct the whole tree of mutations: in particular, for a singularity content $(n, \mathcal{B})$, n represents the number of mutations available. Given that the algorithm has already identified the polygons that are mutation equivalent linked by 7 mutations or less, we will only have to worry about the cases ≥ 8. For our specific basket of singularities, it turns out that there is only one mutation class for every such n, thus all of the classes for polygons with one mutable edge are found.
     For the case of 2 mutable edges, [Pri1] tells us that the polygon is of finite mutation type, and two such polygons are linked by at most 5 mutations. Therefore, since the algorithm has already checked up to distance 8, the listed polygons with 2 mutable edges represent distinct mutation classes.

   - Standard quantum period: the remaining polygons have more than 2 mutable edges, thus we can check whether two such polygons are mutation equivalent by comparing the two quantum periods: if they are related by a $GL_k(\mathbb{R})$ transformation, then they will belong to the same mutation class (see [ACHK15]).

By putting together all these consideration, we have:

Theorem 3.1.1. *There are 69 mutation classes of LDP polygons with singularity content* $(n, k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2))$, *and they are listed in Tables 5.2.*

As for toric surfaces mutation equivalence corresponds to qG–deformation equivalence, the listed polygons describe all the possible degeneration classes for specified singularity content.

In particular, from this classification it is possible to determine the cascade structures for the toric cases, which we will talk about in more detail in Chapter 4.

3.2 T–varieties and Deformations

In this section we will descibe methods to determine which toric surfaces admit a complexity 1 equivariant qG-deformation, relating them to the minimal surfaces if possible. Ultimately we will find all of the possible qG-deformation classes for the given basket of singularities.

3.2.1 T–varieties and Toric downgrades

We will recall some basic definitions about T–varieties and their relation with toric surfaces. For a detailed account about T–varieties, see [AIPSV12].

Definition 3.2.1. A variety X of dimension n is called **T–variety** if it is normal with an effective action of a torus $T = (\mathbb{C}^\times)^k$. The difference $n-k$ is called *complexity* of the variety; in particular, complexity 0 identifies toric varieties.

We are interested in surfaces with a 1–dimensional torus action, namely surfaces of complexity 1.
The main ingredient to understand such varieties is the GIT quotient ([KSZ91]) $Y := X/(\mathbb{C}^\times)$, i.e. a smooth projective curve that encodes geometric information about X.
Let N be a n-dimensional lattice, $M = \mathrm{Hom}(N, \mathbb{Z})$, its dual. Let $\sigma \subset N_{\mathbb{Q}} = N \otimes \mathbb{Q}$ denote a convex polyhedral cone, $\sigma^\vee \subset M_{\mathbb{Q}}$ its dual, and $\Delta \subset N_{\mathbb{Q}}$ a polyhedron. Then define

$$\mathrm{tail}(\Delta) = \{v \in N_{\mathbb{Q}} \mid v + \Delta \subset \Delta\}$$

as the *tailcone* of Δ, where the sum is intended as the Minkowsi sum. In particular, every polyhedron can be written as the Minkowski sum $\Delta = v + \sigma$, where $\sigma = tail(\Delta)$; thus, with respect to this operation, the polyhedra with tailcone σ form the semigroup $\mathrm{Pol}^+_\sigma(N)$, where σ is the neutral element and $\emptyset \in \mathrm{Pol}^+_\sigma(N)$ is the element in the semigroup such that $\emptyset + \Delta = \emptyset$.

Definition 3.2.2. A **polyhedral divisor** with tailcone σ on a variety Y is formal sum

$$\mathcal{D} := \sum_{Z \in \mathcal{P}} \Delta_Z \otimes Z \in \mathrm{Pol}^+_\sigma(N) \otimes \mathrm{Pic}(Y)$$

where $\mathcal{P} \subset \mathrm{Pic}(Y)$ is the subset consisting of all prime divisors of Y.
If $u \in \sigma^\vee \cap M$, the **evaluation** of $\mathcal{D}$ is the divisor in Y defined as

$$\mathcal{D}(u) := \sum_{Z \in \mathcal{P}} \min_{v \in \Delta} \langle u, v \rangle \ Z \in Pic(Y).$$

The **locus** of $\mathcal{D}$ is the non compact open subset of Y defined as:

$$\mathrm{Loc}\mathcal{D} := Y \setminus (\bigcup_{\Delta_Z = \emptyset} Z)$$

Under mild technical assumptions ([AIPSV12]), we can choose our polyhedral divisor so that the associated graded algebra gives the affine scheme

$$X(\mathcal{D}) := \mathrm{Spec} \left(\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(\mathrm{Loc}\mathcal{D}, \mathcal{O}_{\mathrm{Loc}\mathcal{D}}(\mathcal{D}(u))) \right)$$

which has the structure of an affine normal variety with embedded torus action of $T = (\mathbb{C}^\times)^m$ and of dimension $\dim X = \dim Y + m$.
If $\mathcal{D} = \sum_{Z \in \mathcal{P}} \Delta'_Z \otimes Z$ and $\mathcal{D}' = \sum_{Z \in \mathcal{P}} \Delta'_Z \otimes Z$ are two polyhedral divisors in Y, then we can define the inclusion $\mathcal{D}' \subset \mathcal{D}$ whence for every prime divisor $Z \subset Y$ we have $\Delta'_Z \subset \Delta_Z$, and the intersection $\mathcal{D} \cap \mathcal{D}' := \sum_{Z \in \mathcal{P}} (\Delta'_Z \cap \Delta_Z) \otimes Z$.
If the divisors $\mathcal{D}'$ and $\mathcal{D}$ are proper and $\mathcal{D}' \subset \mathcal{D}$, then the respective graded rings give a dominant morphism between the affine varieties $X(\mathcal{D}') \to X(\mathcal{D})$. Moreover, if the map is an inclusion, we denote it by $\mathcal{D}' \prec \mathcal{D}$ and $\mathcal{D}'$ is a face of $\mathcal{D}$.

Definition 3.2.3. A **divisorial fan** $\mathcal{S}$ is a finite collection of proper polyhedral divisors such that $\mathcal{D} \succ \mathcal{D} \cap \mathcal{D}' \prec \mathcal{D}'$ for any two $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$.
A **slice** S_P of the divisorial fan $\mathcal{S}$ is the polyhedral collection $\mathcal{D}_P$ associated to a point $P \in Y$.
The fan $\mathcal{S}$ is called **complete** if every slice is a proper complete subdivision of the lattice N.

From these conditions, it is possible to define a glueing of the affine T–varieties defined by the polyhedral divisors of a fan via the inclusions:

$$X(\mathcal{D}) \leftarrow X(\mathcal{D} \cap \mathcal{D}') \to X(\mathcal{D}')$$

Ultimately, this allows us to define the variety $X(\mathcal{S})$ from the divisorial fan: the affine open covering is given by the collection $\{X(\mathcal{D})\}_{\mathcal{D} \in \mathcal{S}}$, and for a complete fan the variety $X(\mathcal{S})$ is complete.

From now on, we will consider only case of surfaces of complexity 1, so in particular the variety Y is a curve.

A major example of T–variety is the so–called *downgrade* from a toric variety, i.e. restricting to a subtorus action. In this case, the variety Y is actually $\mathbb{P}^1$, and the divisorial fan $\mathcal{S}$ is inherited from the fan underlying the toric variety. We will investigate this case in more detail for the rest of this section.

Let $N' \cong \mathbb{Z}^2$ be a 2–dimensional lattice, $M' = \mathrm{Hom}(N', \mathbb{Z})$ its dual and let $\Sigma \subset N'$ be a complete fan. We have then the toric variety $X := X(\Sigma)$ defined from this lattice in the classical way: the variety X is projective and admits an effective action of a torus $T' \cong (\mathbb{C}^\times)^2$. We would like to restrict such action to a smaller torus $T \cong \mathbb{C}^\times \hookrightarrow T'$ so that such action defined on the divisorial fan agrees with the one on the toric fan.
To do this, consider an element in the dual lattice $m \in M'$ and take the sublattice of N' defined as $N = N' \cap m^\perp$ (where $m^\perp = \{v \in N' \mid \langle v, m\rangle = 0\}$). Moreover, we can attach to it a cosection $s : N' \to N$ and a projection $p : N'_\mathbb{Q} \to (N'/N)_\mathbb{Q}$ coming from the splitting sequence:

$$0 \to N \to N' \to N'/N \to 0$$

This sublattice defines a subtorus action of $T \subset T'$ on X, and a projective fan Σ_Y, namely the coarsest common refinement of the images of the cones $p(\sigma)$ for all $\sigma \in \Sigma$. Thus, we define the divisorial fan $\mathcal{S} = \{D^\sigma\}_{\sigma\in\Sigma}$ where the polyhedral divisors are determined as follows:

$$\mathcal{D}^\sigma = \sum_{\rho\in\Sigma_Y} s(\sigma \cap p^{-1}(\rho)) \otimes D_\rho \tag{3.1}$$

Notice that, as we are dealing with surfaces of complexity 1, the possible tailcones are $(-\infty, 0], \{0\}, [0, \infty)$.
More explicitly, for $m \in M'$ define the set

$$[m = \lambda] = \{v \in N' \mid \langle v, m\rangle = \lambda\} \tag{3.2}$$

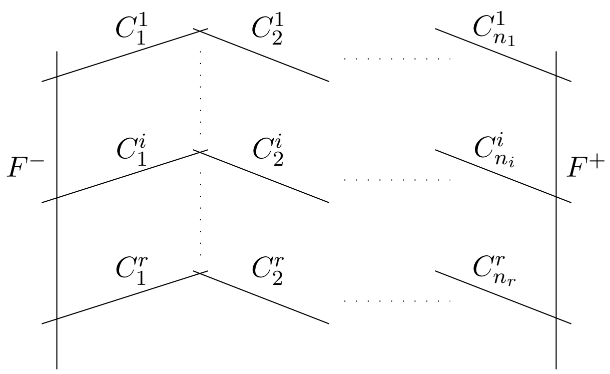
Then, for any cone $\sigma \in \Sigma$ the polyhedral divisor on $Y = TV(\Sigma_Y)$ associated to σ is given by

$$\mathcal{D}^\sigma = s(\sigma \cap [m = 1]) \otimes \{0\} + s(\sigma \cap [m = -1]) \otimes \{\infty\} \tag{3.3}$$

In general, for a complete T–variety the prime divisors correspond to either fixed points of the subtorus action, namely the so–called *horizontal divisors*, or to closed orbits, i.e. polyhedral divisors consisting of the pairs (P, a) where P is a point on Y and a is a vertex of a polyhedron Δ_P (see [AIPSV12]).

Let $\mathcal{S}$ be is a divisorial fan with a polyhedral subdivision for every slice S_i over a finite collection of points $P_i \in Y$, such that the vertices of every polyhedra are labelled as a_k^i for $k = 1..n_i$. Then, we obtain a curve configuration for the surface $X(\mathcal{S})$ in the following way: for every tail cone of the fan in S_i (i.e. $(-\infty, 0]$ and $[0, +\infty)$) we have (at most) two curves $F^{\pm}$ consisting of fixed points of the action of the torus $(\mathbb{C}^{\times})$, while to each vertex of the polyhedral subdivision correspond cuves C_j^i which are the closures of maximal orbits of the said torus action. Every C_j^i intersects a C_k^l if and only $i = l$ and $|m - j| = 1$, while they intersect F^+ (respectively F^-) if and only if a_j^i is a maximal (respectively minimal) boundary point. To sum up, we have:

Lemma 3.2.1 ([AIPSV12])**.** *Let $\mathcal{P} = \{P_1, \ldots, P_r\}$ be a finite set of points on $Y \cong \mathbb{P}^1$ and $\mathcal{S}$ be a complete divisorial fan on it so that at every slice $\mathcal{S}_i$ we have a polyhedral subdivision with vertices $a_j^i = \frac{p_j^i}{q_j^i}$. Then the configuration of curves for $X(\mathcal{S})$ is represented in the following picture*



where the curves have self–intersection numbers:

$$(F^-)^2 = \sum_i a_1^i \qquad (F^+)^2 = -\sum_i a_{n_i}^i \qquad (C_j^i)^2 = -b_j^i \tag{3.4}$$

where b_j^i can be calculated recursively with

$$q_1^i = 1 \qquad \frac{q_j^i}{\overline{p}_j^i} = [b_1^i, \ldots, b_{j-1}^i]$$

and $\overline{p}_j^i = (p_j^i)^{-1}$ (mod q_j^i).

Note that in case either of $F^{\pm}$ does not exist for the fixed torus action, then the configuration is similar by the curves corresponding to maximal and/or minimal boundary points collapse in a unique intersection point.

In the case of a T–variety obtained from the toric downgrade of a smooth toric variety $X = X(\Sigma)$, the divisors defined by projecting the rays of the complete fan are represented by vertices of polyhedra in the divisorial fan in case a corresponds to a point in $N' \setminus N$, and to the curves $F^{\pm}$ in case the vertices lie on the affine line $N_{\mathbb{Q}}$.
As a result, if we consider the Fano polygon of a toric del Pezzo Orbifold X, then the complete fan $\Sigma \in N'$ defining the minimal resolution of such surface gives a complete polyhedral subdivision (for a fixed subtorus action) and thus a curve configuration for the surface representing the minimal resolution of X.

3.2.2 Complexity 1 deformations

We have seen in Theorem 1.2.5 that if two toric surfaces are related by a mutation, then there exists a special pencil linking the two surfaces representing the qG–deformation. In particular, the general fibre of such deformation is a T–variety that inherits the $\mathbb{C}^{\times}$ action from the special fibre. In this section we will discuss how this construction can be adapted to deformation of T–varieties and how this can be linked with deformations of the minimal surfaces found in Chapter 3. For further references on deformations of T–varieties, see [Alt98],[Il09] and [IV11].

Definition 3.2.4. Let Δ be a polyhedron with tailcone $\mathrm{tail}(\Delta) = \sigma$. A **r–parameter Minkowski decomposition** of Δ is a Minkowski sum

$$\Delta = \Delta^0 + \Delta^1 + \cdots + \Delta^r \tag{3.5}$$

such that $\mathrm{tail}(\Delta^i) = \sigma$ for $i = 0..r$.
Such a decomposition is called **admissible** if for every vertex $v \in \Delta$ at most one of the corresponding vertices $v_i \in \Delta^i$ is not a lattice point.

Now, let $Y = \mathbb{P}^1$ be the underlying space for a T–variety of complexity 1, $\mathcal{P} \subset Y$ a finite set of points and $\{\Delta_P\}$ be a collection of polyhedra defining a polyhedral divisor $\mathcal{D}$. The notion of r–parameter Minkowski decomposition is inherited by the polyhedral divisor (and consequently by the divisorial fan): indeed, suppose that for every $P \in \mathcal{P}$ $\Delta_P = \sum_{i=0}^{r_P} \Delta_P^i$ defines a r–parameter Minkowski decomposition of the polyhedron Δ_P, then we call this a decomposition of the polyhedral divisor $\mathcal{D}$ and we say it is admissible

if the decomposition of Δ_P is admissible for every $P \in \mathcal{P}$.
Analogously, for a divisorial fan $\mathcal{S}$, consider the slice $\mathcal{S}_P$ and a polyhedral subdivision $\mathcal{C}$ on it. Then an admissible r–parameter Minkowski decomposition of $\mathcal{C}$ consists of a collection of admissible r–parameter Minkowski decompositions as in 3.5 for every $\Delta \in \mathcal{C}$. Consequently, an admissible r–parameter Minkowski decomposition of the fan $\mathcal{S}$ consists of admissible r_P–parameter decompositions for every $\mathcal{S}_P$.

In the particular case of surfaces X with cyclic quotient singularities, it was proved in [Il09] that admissible Minkowski decompositions of certain polyhedra coming from the local toric nature of such singularities correspond to one–parameter toric deformations of X. In more detail, let $\frac{1}{n}(1,q)$ be a cyclic quotient singularity, and $X_{(n,q)} = TV(\sigma)$ denote the affine toric surface constructed from the cone $\sigma \in N' \cong \mathbb{Z}^2$ representing the quotient singularity. In the fashion of [Reid1], let $[a_1, \ldots, a_l]$ denote the continued fraction expansion of $\frac{n}{n-q}$, such that if $m_i \in M' = (N')^\vee$ are the generators of the semigroup $\sigma^\vee \cap M'$, then $m_{i-1} + m_{i+1} = a_i m_i$. Let $(k_1, \ldots, k_l)$ be a chain of integers and define by induction a sequence $\{\alpha_i\}$ so that

$$\alpha_1 = 0 \qquad \alpha_2 = 1 \qquad \ldots \qquad \alpha_{i-1} + \alpha_{i+1} = k_i \alpha_i. \tag{3.6}$$

Then set

$$K_l(X_{(n,q)}) = \Big\{ (k_1, \ldots, k_l) \in \mathbb{N}^l \mid [k_1, \ldots, k_l] = 0,\ \alpha_i \geq 0 \text{ and } k_i \leq a_i \Big\}$$

consists of elements corresponding to *P–resolutions* of $X_{(n,q)}$ in the sense of [KSB88].

Now, fix two integers h, p with $1 \leq h \leq l$ and $1 \leq p \leq a_h$. In $N_{\mathbb{Q}}$ (using the notation of 3.2), define the line $H^h = [m_h = 1]$ which contains a lattice point as m_h is a minimal generator. Considering this point as the origin of an affine line, the space H^h inherits a lattice structure L^h from N', and the set $Q := H^h \cap \sigma$ has the structure of a polytope, more specifically of an interval (β, γ).
Finally, if d is chosen such that $0 \leq pd \leq |Q|$, an admissible Minkowski decomposition of the form

$$Q = [\beta, \gamma - pd] + p \cdot [0, d]. \tag{3.7}$$

Theorem 3.2.1 ([Il09] Theorem 2.2, Theorem 3.2, Proposition 4.1).

1. *A Minkowski decomposition of type 3.7 gives a nontrivial one–parameter toric deformation $\pi^d_{h,p} : \mathcal{X} \to \mathbb{A}^1$. Every one–paramenter toric deformation come from*

such decompositions for fixed p, h.

2. *A Minkowski decomposition of type 3.7 corresponds to an element $\boldsymbol{k} \in K_l$ if and only if $a_h - k_h \geq pd \geq 1$. Moreover, if $X_{(n,q)}$ is a T–singularity then there exists a $\boldsymbol{k} \in K_l(X_{(n,q)})$ such that $a_i = k_i$ for $i \neq h$ and $a_h = k_h + b$, with $b \in \mathbb{N}$.*

3. *The general fibre of the deformation $\pi^d_{h,p}$ has a $[a_1, \ldots, a_h - pd, \ldots, a_l]$ singularity at the origin and A_{d-1} singularities in d other points (A_0 is defined to be a smooth point).*

Remark 3.2.1. In particular, a Minkowski decomposition gives a smoothing of the cyclic quotient singularity when $d = 1$ and $p = a_h - 1$. Note that this condition is necessary, but not sufficient. Moreover, from Part (2) in the theorem above, $\pi^1_{h,p}$ corresponds to a qG–smoothing.

If we consider a toric downgrade of a toric del Pezzo orbifold, then every admissible decomposition of the divisorial fan associated to it comes from admissible decompositions of the polyhedra in the slices $\mathcal{S}_0$ and $\mathcal{S}_\infty$. Indeed, by fixing a subtorus action $m \in M$, the projections of the (possibly singular) cones on the level lines $[m = \pm 1]$ give polyhedral subdivisions on the slices which can then be decomposed in the fashion of 3.7 (with $d = 1$). Now, we know that single cones admit a one–parameter deformation (and possibly a smoothing) of the cyclic quotient singularity of the cone.

In terms of polyhedral divisors, [IV11] explains how to construct the families of divisors and fans by constructing a total family Y^{tot} over a base space. More specifically, if B is an affine variety where 0 is cut out by the regular sequence $t_1, \ldots t_k$, for $0 \leq j \leq k$ define B_j the subvariety cut out by $t_{j+1}, \ldots, t_k$ and $Y^{tot}_j := V(t_{j+1}, \ldots, t_k) \subset Y^{tot}$. The family $\gamma : Y^{tot} \to B$ is such that $Y^{tot}_j = \gamma^{-1}(B_j)$ and in particular $\gamma^{-1}(0) = Y$. On Y^{tot} we then consider a family of pairwise different prime divisors $D^{tot}(P, i)$ that intersect the subvarieties Y^{tot}_j properly. In particular, $D^{tot}(P, i)|_Y = P$. The polyhedral divisor on the total family Y^{tot} is then defined as follows:

$$\mathcal{D}^{tot} = \sum_{P,i} \Delta^i_P \otimes \mathcal{D}^{tot}(P, i) \tag{3.8}$$

so that, for each $P \in Y$, $\sum_i \Delta^i_P = \Delta_P$. Thus, $\mathcal{D}^{tot}|_Y = \mathcal{D}$ and $\mathcal{D}^{tot}(u)|_Y = \mathcal{D}(u)$ for every $u \in \sigma^\vee \cap M$.

Since we are dealing with toric surfaces with cyclic quotient singularities, we have seen that for every decomposition of the projection of every cone on the level lines we have a one–parameter deformation giving a family $\pi : \mathcal{X} \to \mathbb{A}^1$. Thus we want to find a construction to suitably glue together these deformations in order to adapt them to the divisorial fan inherited from the toric downgrade.

Let $\mathcal{P} \subset Y$ be a finite set of points, $y_P \in \mathbb{C}(Y)$ a rational function with zero in P for every $P \in \mathcal{P}$, and $t_{P,1}, \dots, t_{P,r_P}$ coordinates on $\mathbb{A}^{r_P}$ (note $t_{P,0} = 0$). Take $B \in \prod_P \mathbb{A}^{r_P}$ to be an open affine neighbourhood of the origin, so that a divisor of $\mathbb{P}^1 \times B$ is of the form $V(y_P - t_{P,i})$. The prime polyhedral divisors on the total family are then given by $\mathcal{D}^{tot}(P, i) = V(y_P - t_{P,i})$.
For $\lambda \in B$, let $Y^{tot}_\lambda \cong Y$ be the fibre over the point λ; we can then define the polyhedral divisor $D^{(\lambda)}$ as the restriction of $\mathcal{D}^{tot}$ to Y^{tot}_λ, so in particular it is a polyhedral divisor on Y. For each $P \in \mathcal{P}$ let λ be defined by the regular sequence of equations $t_{P,i} - \lambda_{P,i}$ (with $\lambda_{P,0} = 0$), and for $0 \le i \le r_P$ let $D^{(\lambda)}(P, i)$ denote the prime divisor defined by $V(y_P - \lambda_{P,i})$ on Y. Then, the polyhedral divisor $D^{(\lambda)}$ on the fibre is explicitly given by

$$\mathcal{D}^{(\lambda)} = \sum_{\substack{P \in \mathcal{P} \\ 0 \le i \le r_P}} \Delta^i_P \otimes \mathcal{D}^{(\lambda)}(P, i) \tag{3.9}$$

Now, consider every $\mathcal{D}$ as an element of the fan $\mathcal{S}$ with an admissible polyhedral decomposition $\mathcal{D}_P = \sum_{i=0}^{r_P} \Delta^i_P$ for every $P \in \mathcal{P}$ which consequently give an admissible decomposition of the fan. To obtain a description of the divisorial fan on Y^{tot}, for every subset $\mathcal{I} \in \mathcal{S}$ and $\lambda \in B$ define

$$\mathcal{D}^{\mathcal{I}} = \bigcap_{\mathcal{D} \in \mathcal{I}} \mathcal{D} \qquad \mathcal{D}^{\mathcal{I},tot} = \bigcap_{\mathcal{D} \in \mathcal{I}} \mathcal{D}^{tot} \qquad \mathcal{D}^{\mathcal{I},(\lambda)} = \bigcap_{\mathcal{D} \in \mathcal{I}} \mathcal{D}^{(\lambda)} \tag{3.10}$$

Finally, it is possible to define the divisorial fans on the total space and on the fibres over λ:

$$\mathcal{S}^{tot} = \{\mathcal{D}^{\mathcal{I},tot}\} \qquad \mathcal{S}^{(\lambda)} = \{\mathcal{D}^{\mathcal{I},(\lambda)}\} \tag{3.11}$$

In this setting ($Y \cong \mathbb{P}^1$ and the Minkowski decompositions underlying $\mathcal{D}^{tot}$ are admissible), we have the following

Theorem 3.2.2 ([IV11], Theorem 2.8, Theorem 4.4)**.** *The map $\pi : X(\mathcal{D}^{tot}) \to B$ gives a flat family such that $\pi^{-1}(0) = X(\mathcal{D})$, $\pi^{-1}(\lambda) \cong X(\mathcal{D}^{(\lambda)})$ and its embedding $X(\mathcal{D}) \hookrightarrow X(\mathcal{D}^{tot})$ is induced by $Y \hookrightarrow Y^{tot}$.*
The map $\pi : X(\mathcal{S}^{tot}) \to B$ gives a flat family such that $\pi^{-1}(0) = X(\mathcal{S})$, $\pi^{-1}(\lambda) = X(\mathcal{S}^{(\lambda)})$ and its embedding $X(\mathcal{S}) \hookrightarrow X(\mathcal{S}^{tot})$ is induced by $Y \hookrightarrow Y^{tot}$.

3.3 Complexity 1 qG-deformations of toric surfaces

In this section, we come back to the idea of Theorem 1.2.5 and relate it to the deformation of T–varieties discussed in section 4.3. Specifically, we would like to see the minimal surfaces found in Chapter 3 as T–varieties (or deformations thereof) linked to mutation classes as found in Chapter 2.
Theorem 3.3.3) is part of the work in progress together with Edwin Kutas ([CK]).

Firstly, we recall that for a toric surface a mutation is defined as follows: let $P \subset N'_{\mathbb{Q}}$ (where, using the notation of the previous section, $N' \cong \mathbb{Z}^2$) be a Fano polygon underlying a toric surface X. Let $m \in M' = (N')^{\vee}$ be a primitive vector representing an inner normal vector for an edge E of P, and let m_{max} and m_{min} be the maximal and minimal values of m on P.
We give an orientation to the polygon P so that the vertices are labelled starting from v_1 being a vertex at height m_{max} and so that the edge $E = [v_i, v_{i+1}]$ is at height m_{min}. Notice that the choice of orientation does not affect the mutation.
Suppose the lenght of E is given by $|v_{i+1} - v_i| = df$, where $d, f \in \mathbb{N}$. Then to mutate the polygon P in P' we fall in one of the following cases:

1. There is only one vertex at height m_{max}: then say P has $h - 1$ vertices and the mutation adds an extra vertex v_h to P';
2. There is another vertex at height m_{max}: then say P has h vertices and there is an edge at height m_{max}, namely $[v_1, v_h]$.

So, by fixing the integers d, f, a mutation P' of P is given by the vertices:

$$v'_j = \begin{cases} v_j & 1 \leq j \leq i \\ v_j + \langle m, v_j \rangle f & i < j \leq h \\ v_h + m_{max} f & j = h-1 \end{cases} \tag{3.12}$$

Theorem 3.3.1. *Let X_1, X_2 be two toric orbifold del Pezzo surfaces corresponding to the two Fano polygons P_1, P_2. Furthermore, assume that the two polygons are mutation equivalent , so there exists a qG–deformation family $\pi : \mathcal{X} \to B$ such that for $\lambda_1, \lambda_2 \in B$ then $\pi^{-1}(\lambda_1) = X_1$ and $\pi^{-1}(\lambda_2) = X_2$. Then the general element X' of the family is a T–variety corresponding to an equivariant blow up of a toric surface $\overline{X}$. Moreover, the toric surfaces X_1, X_2 are obtained from $\overline{X}$ via toric blow ups.*

Proof. We have seen how taking an admissible Minkowski decomposition of a polyhedral divisor of a toric downgrade gives a one–parameter deformation of cyclic quotient singularities ([IV11], [Il09]).
So, start with a Fano polygon $P_1 \subset N'_{\mathbb{Q}}$ defining a toric fan Σ_1 so that $X_1 = X(\Sigma_1)$ is the toric variety defined by it. Fix a subtorus action $m \in M'$ such that m defines the inner normal vector of a mutable edge. Suppose now the polygon P_1 is as in Case 1 (Case 2 is analogous), so $\langle v_1, m\rangle = m_{max}$, and label the vertices in a counterclockwise fashion. By eventually applying an $SL_2(\mathbb{Z})$ transformation, we can move the vertices of P_1 (without changing its structure) so that we can group the vertices as follows: if (x_j, y_j) are the coordinates in $N'_{\mathbb{Q}}$ for v_j, then

$$(x_j, y_j) = \begin{cases} x_j < 0, y_j \geq 0 & 1 \leq 1 \leq k \\ x_j \leq 0, y_j < 0 & k < j \leq i \\ x_j > 0, y_j \leq 0 & i < j \leq l \\ x_j \geq 0, y_j > 0 & l < j \leq h-1 \end{cases}$$

as represented in the example on the left side of Figure 3.1 below. On the right side, we can see the subdivisions of the divisorial fan $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_\infty$ given by the chosen $(\mathbb{C}^\times)$ action $m \in M'$. The points a_j denote the projections of the vertices on the polygon to the level lines $[m = \pm 1]$ representing the subtorus action: namely, in the notation of 3.3, for a polyhedral divisor $\mathcal{D}^\sigma$ (where σ is the cone generated by the vertices v_j, v_{j+1}) they represent the vertices of the polyhedra defined by the projection $s(\sigma \cap [m = \pm 1])$.

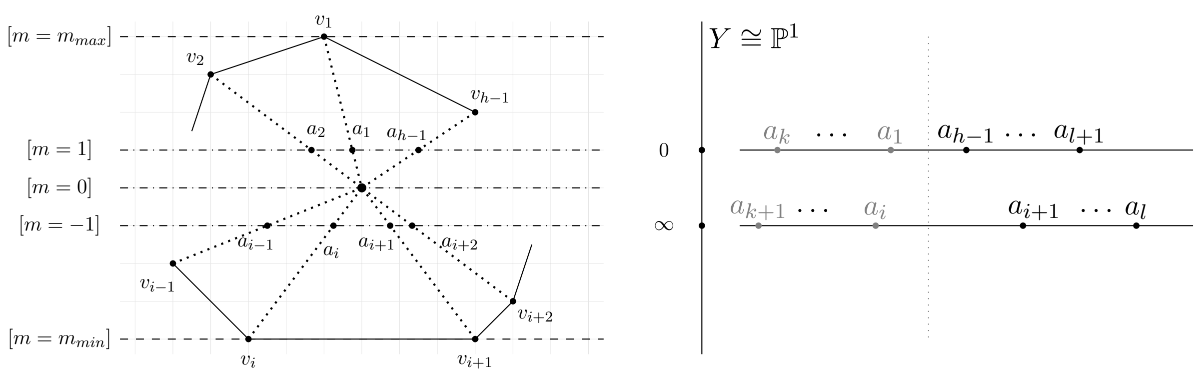


Figure 3.1: Fano polygon P_1 and decomposition as T–variety

By applying a mutation as described in 3.12, we obtain the Fano polygon $P_2 \in N'_{\mathbb{Q}}$ with the extra vertex v'_h (and its projection on the slice $\mathcal{S}_0$), as pictured in the example in Figure 3.2 below. The polygon defines a fan Σ_2 so that the toric surface defined by such a fan $X_2 = X(\Sigma_2)$ is the mutated surface (and thus a qG–deformation of X_1). In particular, the points in grey are the vertices that remain unchanged and the boxed one is the extra vertex added after the mutation.

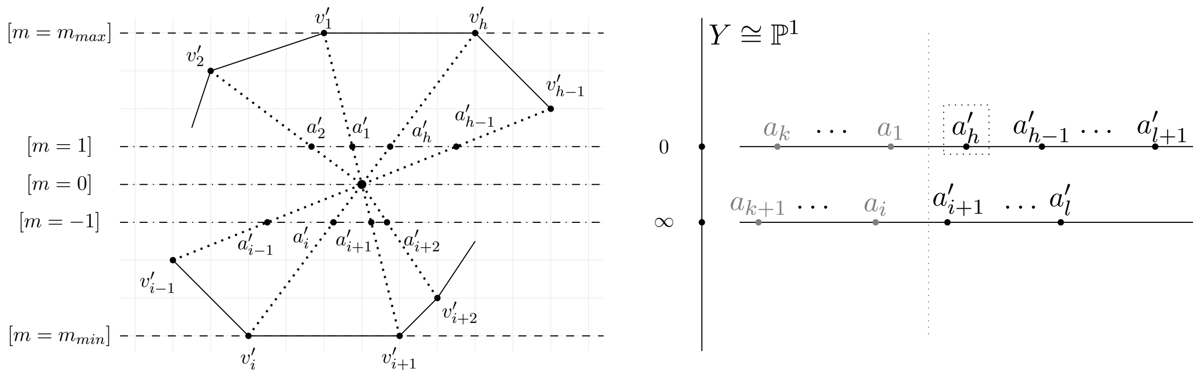


Figure 3.2: Fano polygon P_2 and decomposition as T–variety

So the vertices $a_1, \ldots, a_k$ and $a_{k+1}, \ldots, a_i$ (i.e. on the left hand side of the grey line dividing the polyhedral fan) are unchanged. On the right hand side, they are translated as follows: as described in 3.12, there exists an integer f such that $|v_{i+1} - v_i| = df$, with $d \geq m_{min}$. Thus

$$a'_j = \begin{cases} a_j & 1 \leq j \leq i \\ a_j - f & i+1 \leq j \leq l \\ a_j + f & l+1 \leq h-1 \\ a_1 + f & j = h \end{cases} \tag{3.13}$$

For the slice $\mathcal{S}_0$, we have the polyhedral subdivision given by the intervals:

$$(-\infty, a_k], [a_k, a_{k-1}], \ldots, [a_2, a_1], [a_1, a'_h], [a'_h, a'_{h-1}], \ldots, [a'_{l+2}, a'_{l+1}], [a'_{l+1}, +\infty) \tag{3.14}$$

and we have the following admissible decompositions:

$$\begin{gathered} (-\infty, a_k] = (-\infty, a_k] + \{0\} \\ [a_j, a_{j-1}] = [a_j, a_{j-1}] + \{0\} \\ [a_1, a'_h] = [a_1, a_1 + f] = [a_1] + f \cdot [0, 1] \\ [a'_j, a'_{j+1}] = [a_j + f, a_{j+1} + f] = [a_j, a_{j+1}] + f \cdot \{1\} \quad l+1 \leq j \leq h-1 \\ [a'_{l+1}, \infty) = [a_{l+1} + f, \infty) = [a_{l+1}, \infty) + f \cdot \{1\} \end{gathered} \tag{3.15}$$

giving an admissible decomposition for the slice $\mathcal{S}_0$, while we fix the polyhedral subdivision on $\mathcal{S}_\infty$ by considering the trivial decomposition $\mathcal{S}_\infty = \mathcal{S}_\infty + \emptyset$.
These admissible decompositions correspond to decompositions of the cones as described in 3.7, where $p = f$ and d is fixed so that $fd \leq |v_{i+1} - v_i|$.
Thus, by Theorem 3.2.1, they define a one–parameter deformation of such cone, and by 3.2.2 (as we effectively have an admissible decomposition of the whole fan), it corresponds to a flat family $\pi : \mathcal{X} \to \mathbb{A}^1$, and the divisorial fan of the general element of such deformation is represented in Figure 3.3 below.

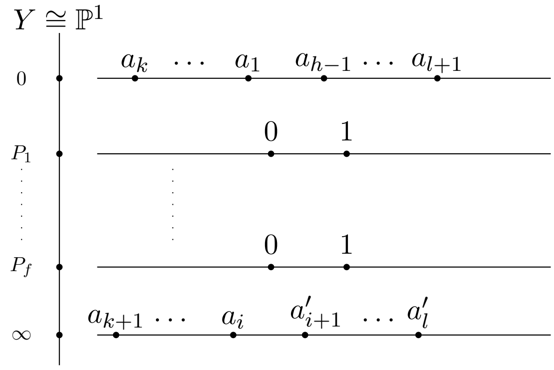


Figure 3.3: Divisorial Fan for the generic element in the deformation family

By Lemma 3.2.1, this corresponds to blow ups in f general points of a toric surface $\overline{X} = X_Q$ (where Q is a Fano polygon) on the curve F^+ corresponding to one of the curves fixed by the subtorus action. Specifically, the toric surface $\overline{X}$ inherits the subtorus action from $X_1 = X_P = X(\Sigma_1)$ and is represented by the divisorial fan in Figure 3.4 below.

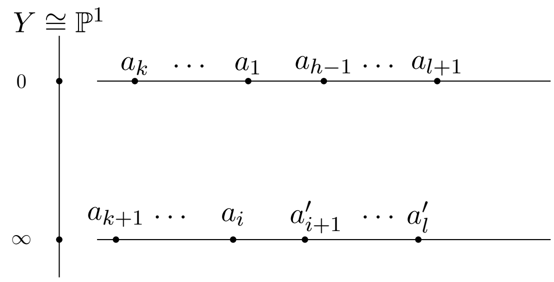


Figure 3.4: Divisorial Fan for $\overline{X}$

To conclude, the fibre over the general point of the mutation linking $X(\Sigma_1)$ and $X(\Sigma_2)$ is a T–variety $X' = X(\mathcal{S})$, where the divisorial fan $\mathcal{S}$ represents the blow up at f general points of a toric surface $X(\overline{\Sigma})$ on the line fixed by the subtorus action.

□

Remark 3.3.1. Using analogous polyhedral decomposition, it is possible to also have blow ups of the curve F^- (fixed by the torus subaction) by considering decompositions in intervals $[-1, 0]$. As a result, this construction works for configurations having blow ups in general points on both curves F^- and F^+ of $X(\overline{\Sigma})$

Generically speaking, these blow ups change the singularity type of the cones that get blown up from Q. More specifically, the polygons P_1, P_2 represent blow ups of Q at toric points, meaning the vertices of the Fano polygon we added to get P_1, P_2 will give new cones that can possibly change the rigid content of the surface $\overline{X}$.
Indeed, if we look at the curve configurations of the minimal surfaces in Chapter 3, it is clear that the curves coming from the resolution of the singularities come from blow ups of (-1) and/or (-2)–curves, and most importantly they are blow ups of toric surfaces. Therefore, our idea is to find a suitable subtorus action for the underlying toric surfaces so that the resulting surface fits in the structure above, and is thus the midpoint of a mutation.

Definition 3.3.1. Let $\overline{X} = X_Q$ a toric del Pezzo orbifold. Suppose a surface S is obtained by blowing up two curves $F^{\pm} \in Q$ in (respectively) $f_{\pm}$ general points. Then the surface S is of one of the following types:

Type 0 if the surface S is toric, i.e. the curve configuration of the resolution is a regular polygon;

Type 1 if the two curves $F^{\pm}$ are fixed by the same subtorus action for a choice of $m \in M'$, i.e. the curve configuration looks like in Figure 3.5;

Type 2 if the two curves $F^{\pm}$ are fixed by two different subtorus actions; these configurations appear as compositions of Type 1 constructions.

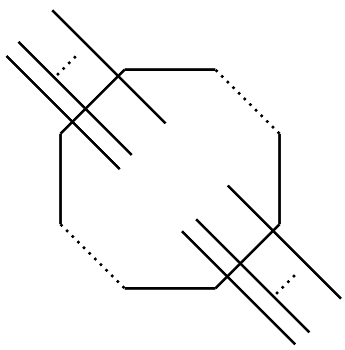


Figure 3.5: Curve configuration of Type 1

Now, suppose S is one of the minimal surfaces listed in Theorem 2.8.1, and let $\psi : S \to \overline{X}$ be the blow–down of the (-1)–curves so that $\overline{X}$ is a toric surface.
Let Q be the Fano polygon of such surface $\overline{X}$, then we can divide the listed surfaces in the three aforementioned classes.

Propostion 3.3.1. *The surface $S^{3,1}_{(1,1)}$ has a configuration that is isomorphic to a Type 1 configuration.*
The surfaces $S^{2,1}_{(2,2)}, S^{2,2}_{(2,2)}, S^{3,1}_{(0,3)}, S^{3,2}_{(0,3)}$ have configurations that are isomorphic to Type 2 configurations.

Proof. It is sufficient to see that the curves from which we take the blow up belong to a linear system that also contains curves on the toric boundary. Thus, we can choose the blow ups to be on these curves and obtain isomorphic configurations.
For instance, the configuration for the surface $S^{3,1}_{(1,1)}$ is obtained by blowing up two infinitely near points on a curve of self intersection (-2), as in Figure 3.6 below. The first blow up is going to be on a (2)–curve in a linear system, and the two adjacent curves of self intersection 1 and (-1) that oare on the toric boundary also belong to such linear system. Thus, the second blow up can be taken on the (-1) curve on the toric boundary, giving then a configuration of Type 1 that is isomorphic to the initial configuration.

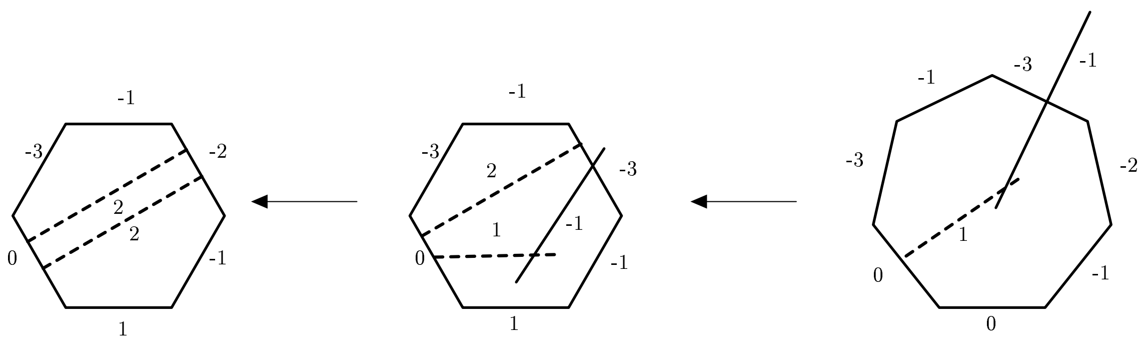


Figure 3.6: $S^{3,1}_{(1,1)}$ as Type 1 configuration

In similar fashion, for case $S^{2,1}_{(2,2)}$ the configuration is obtained by blowing up two infinitely near points on a (-2)–curve on the toric boundary. The first blow up is going to be on a (0)–curve in a linear system, so that the two adjacent (-1)–curves in the configuration also belong to it. Thus, for the second blow up we can choose a point on the (-1)–curve intersecting the (-3)–curve on the toric boundary, ultimately obtaining a Type 2 configuration.

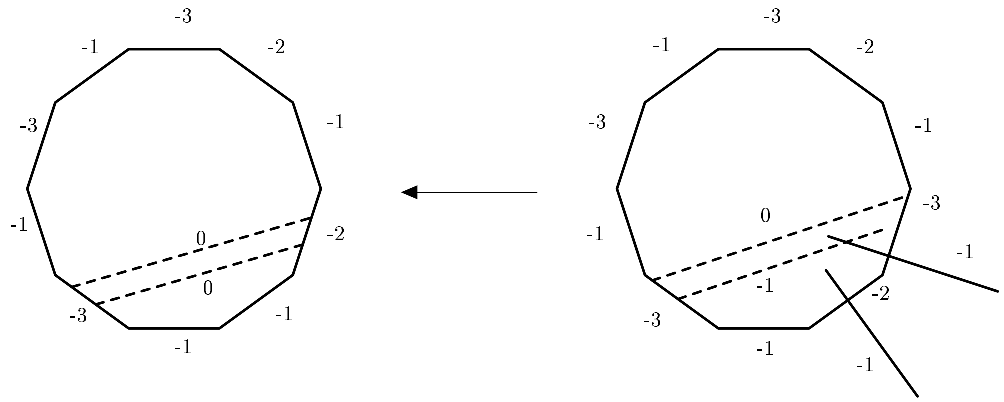


Figure 3.7: $S^{2,1}_{(2,2)}$ as Type 2 configuration

$\square$

Corollary 3.3.1. *For the configurations of surfaces in Theorem 5.1, we have*

	# surfaces
Type 0	10
Type 1	18
Type 2	5

If we are in the situation of Type 0, then there is nothing to prove as the general element of the deformation family is toric, thus it will have a model in the list in Chapter 6.2.

Suppose now S is a surface of Type 1. Then Q has only smooth cones or rigid singularities except for (possibly) the cones for which the exceptional curves coming from the resolutions of singularities are represented by points on the invariant lines. Then, by blowing up these points, we change the singularity type of this cone until we reach the singularity type we are looking for (specifically, either $\frac{1}{3}(1,1)$ or $\frac{1}{5}(1,2)$).

Theorem 3.3.2. *Let S be a nontoric minimal del Pezzo orbifold with basket of singularities $\mathcal{B} = \{k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2)\}$ and $h^0(-K_S) \neq 0$. Let $\varphi : Y \to S$ be the minimal resolution, and assume that the curve configuration is of Type 1 (as in Figure 3.5), obtained by blowing up a toric surface in $f = f_- + f_+$ general points on the curves F^- and/or F^+, fixed by a subtorus action. Then there exists a toric surface $X(\Sigma)$ and a qG–degeneration $\pi : \mathcal{X} \to B$ where $\pi^{-1}(0) = X$ and general fibre $\pi^{-1}(\lambda) = S$.*

Proof. Our aim is to reconstruct the deformation family as shown in Theorem 3.3.1. Thus, consider the toric surface $X(\overline{\Sigma})$ corresponding to the curve configuration obtained from blowing down the $f = f_{-} + f_{+}$ curves of S. As the surface S is minimal, then its curve configuration does not contain any floating (-1) curve or any chain of curves coming from the resolution of a T–singularity; thus, except for the $F^{\pm}$ curves, the configuration will consist of sequences of $[-3]$ and $[-3, -2]$ curves separated by curves with self intersection ≥ -1.
The self–intersection of the curves $F^{\pm}$ will then be ≤ -2, so the underlying toric surface will be a surface with basket of singularity of type $\mathcal{B}$. To find the toric surfaces linked to the mutation, we have to fix a subtorus action of the surface so that the non–toric blow ups will give the polyhedral fan of the general element of a mutation. As we are in the configuration of Type 1, then there exists a subtorus action fixing $F^{\pm}$. Hence, by possibly considering an $SL_2(\mathbb{Z})$ transform, we can assume the Fano polygon Q of $\overline{X}$ has a subtorus action associated to $m = (1, 0)$ so that the two interior points of Q representing the curves $F^{\pm}$ lie in the affine line $[m = 0]$.
Without loss of generality, suppose we fix a subtorus action so that the fixed curve $F^{\pm}$ is represented by the vertex $\overline{v} = (1, 0) \in N'$ (the case for F^{-} is analogous). If we look at the polygon Q, there are not many possibilities for the cone containing the point $\overline{v}$ as F^{+} can be either a (-2) or a (-1)–curve coming from blow downs of (-1)–curves intersecting $[-3, -2]$ or $[-3]$ (see Theorem 2.3.1). Consequently, when considering the divisorial fan inherited from this subtorus action, there are only few possibilities for the vertices a'_l, a'_{l+1} (in the notation of Figure 3.4).
Similarly to the proof of Theorem 3.3.1, a non–toric blow up of such surface corresponds to adding an extra point on Y and a correspoding slice in the divisorial fan so that the polyhedral divisors behave accordingly.
Thus, by using the construction of the said proof, we can reconstruct the divisorial fan for the degeneration of S.

□

Corollary 3.3.2. *Every minimal surfaces of Type 1 listed in Table 2.5 admits a degeneration to a toric surface.*

It remains to show that the surfaces with Type 2 configurations admit degenerations to toric surfaces. Indeed, for our minimal models Type 2 configurations consist of blow ups of curves on the toric boundaries fixed by distinct subtorus actions.
Thus the next step is to check that the complexity 1 equivariant deformations defined locally by the admissible Minkowski decompositions can be glued together.

Theorem 3.3.3. *Let S be a nontoric minimal del Pezzo orbifold with $h^0(-K_S) \neq 0$ and basket of singularities $\mathcal{B}$ as above. Let $\varphi : Y \to S$ be the minimal resolution and assume that the curve configuration on Y is of Type 2, i.e. consisting of blowing up a toric surface in f^i general points on a finite number of curves F_i fixed by distinct subtorus actions. Then the surface X represents a smoothing of a toric surface with same singularity content.*

Proof. From theorem 3.3.2 we have seen that blowing up curves fixed by a subtorus action gives a surface that is a general element of a qG–deformation of toric surfaces. For Type 2 configurations we have to show how the deformation given by glueing two such deformations gives a degeneration to a toric surface. To this end, we will rely on the construction of the so–called *focus–focus* singularities in relation to smoothing of toric varieties as described by the Gross–Siebert program [GS11].
Indeed, following the work of Prince ([Pri18]), it is possible to construct an affine manifold with singularities that serves as a partial smoothing of the boundary of the Fano polygon P associated to the toric variety X_P. More specifically: for $P \subset N'_{\mathbb{Q}}$ Fano polygon, the dual polygon $P^\vee$ is the base space for a *special Lagrangian torus fibration* given by the moment map (see [Ful93]). This is used to contstruct a smoothing of the T–singularities of the variety X_P: as a basic example, consider an A_n singularity. In M this is, up to an $SL_2(\mathbb{Z})$ transformation, defined by the cone $(-1, n+1)$, $(0,0)$, $(1,0)$. Let us denote the cone as an affine piece by B_n. In terms of the moment map the fiber over $(0,0)$ is a point and is the A_n singularity. We now smooth this singularity out. Our general fiber will look like the following: there are two affine charts

$$\begin{aligned} U_1 &= B_n - \{(0,t) \,|\, t \geq t_{\text{height}}\} \\ U_2 &= B_{n-1} - \{(0,t) \,|\, t \leq t_{\text{height}}\} \end{aligned} \tag{3.16}$$

The point t_{height} is called a focus–focus singularity with monodromy one. The singularity of this fiber corresponds to the pinched torus. The singular point of this fiber correspond to the singular point of the fiber of the smoothing of the A_n singularity. We now see that the singularity in the corner is determined in the chart with an A_{n-1} singularity, we can repeat this until we obtain a complete smoothing.
In [Pri18], the author showed how to each singularity we can associate a monodromy invariant line: by moving the focus–focus singularities along it we obtain a qG–smoothing of each singualrity. Via the moment map, we can repeat this contstruction for all of the singularities in X_P and obtain an integral affine manifold $P^\vee \in M'$ so that for every smoothable singularity we assiciate a monodromy invariant line with as many focus–focus points as needed. In this way we can construct a total smoothing of the surface X_P. Proposition 9.8 in [Pri18] assures that the smoothing is qG.

On the other hand, adding a focus–focus singularity along a monodromy invariant line is equivalent to blowing up a general point on a torus invariant curve. In particular, if our monodromy invariant line is cut out by a chosen subtorus action $m \in M'$, then the we are blowing up a general point on the torus invariant curve corresponding to $v \in N'$ where $v \in [m = 0]$. Therefore, by subsequently chosing the subtorus actions and adding a focus–focus singularity on the respective monodromy invariant curve we can reconstruct a configuration of Type 2.

Finally, we are left with checking that the Type 2 configurations arise from an integral affine manifold representing the base of the special Lagrangian torus fibration.
This is ensured by the existence of *Looijenga pairs* as defined in [GHK15] for our configurations in the minimal resolutions. More precisely, let Y be the minimal resolution of a log del Pezzo S such that the configuration of curves is not necessarily toric. Then it is possible to choose an element $D \in |-K_Y|$ such that $D = D_1 + \cdots + D_k$, where every D_i represents a subset of rational curves in $\mathrm{Z}_1(Y)$. In particular, D is either an irreducible rational nodal curve, or a cycle of $k \leq 2$ smooth rational curves. The pair (Y, D) is the aforementioned Looijenga pair. Roughly speaking, to construct an integral affine manifold we induce a toric–like construction from the configuration D on a fan Σ in $\mathbb{R}^2$: by trying to glue the cones resulting from D, we obtain an object that does not represent a Fano polygon. Indeed, it is not supported on $N' \cong \mathbb{Z}^2$, but instead it is supported on a cone so that the usual toric relations on self intersections of curves hold. Locally the construction is toric, but the glueings are not torus invariant.
This construction associates to (Y, D) a pair (B, Σ) where B will inherit the structure of integral affine manifold with a singularity at the origin and Σ will represent the decomposition of B into cones.

As a result, assume we start with a surface $\overline{S}$ such that its minimal resolution $\overline{Y}$ has a configuration of Type 1. For simplicity, assume $\overline{Y}$ is obtained by blowing up a toric surface in f^1 general points on a curve F^1 fixed by a torus action $m \in M'$ (the case of two curves is analogous). Then we can recover the polygon $P^\vee$ as an integral affine manifold with focus–focus singularities $S^1_1, \dots, S^1_{f^1}$ placed on the respective monodromy line. To obtain a configuration of Type 2 we need to add more focus–focus singularities on the monodromy lines linked to subtorus actions fixing other curves F_i of the configuration in Y. For each S^1_j $j = 1, \dots, f^1$ we can consider the degeneration sending the point S^1_j to the boundary of $P^\vee$. In terms of 3.3.1, this corresponds to taking the degeneration of the T–variety $X(\mathcal{S})$ to one of the mutation equivalent toric surfaces, say X_1, and fix a new subtorus action on the Fano polygon P_1 of X_1.
By inductively applying these steps we obtain a Fano polygon P representing the degen-

eration of the nontoric variety for the configuration of curves of the minimal resolution is represented by the polygon $P^\vee \in M'_{\mathbb{Q}}$ with $\sum_i f^i$ focus–focus points.
Hence, the surface S is a complete smoothing of the toric surface X.

$\square$

Corollary 3.3.3. *Every minimal surfaces of type 2 listed in Table 2.5 admits a degeneration to a toric surface.*

3.4 Examples

Example 1: $S^3_{(0,1)} \rightsquigarrow \mathbb{P}(1,2,5)$

Consider one of the minimal surfaces obtained from running the Directed MMP described in Theorem 2.3.1 for a del Pezzo Orbifold with $\frac{1}{5}(1,2)$ singularity, namely $S^3_{(0,1)}$. The configuration of curves coming from the MMP presents a (-1)–curve resulting from a non toric blow up of a (-1)–curve of a toric surface $\overline{X}$ as pictured in Figure 3.8 below.

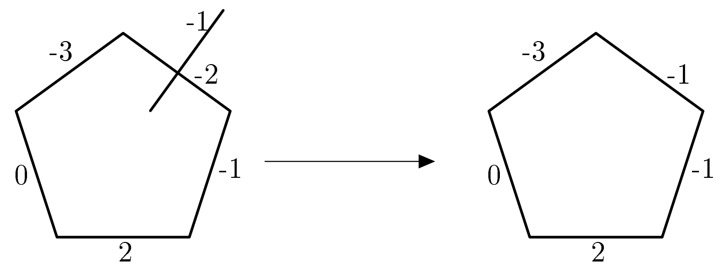


Figure 3.8: Type 1 Non toric blow up of toric curve configuration

The surface $\overline{X}$ is represented by the Fano Polygon $\overline{P} \in N' \cong \mathbb{Z}^2$ in Figure 3.9: the interior points of the polygon are labelled with the self–intersection of the curves they represent and the cones generated by such polygon are represented by the dotted subdivisions, and they are labelled by σ_i. Moreover, we have made a choice of a subtorus action on $\overline{X}$, namely $m = (1,0) \in M' \cong \mathbb{Z}^2$, and the dash–dotted lines represent the level lines

$$\begin{aligned} [m = 1] &= V(x = 1) \subset N'_{\mathbb{Q}} \\ [m = -1] &= V(x = -1) \subset N'_{\mathbb{Q}} \end{aligned} \tag{3.17}$$

and the cosection $s : N' \to N$ is chosen to be $s(x,y) = y$. As discussed in 3.3, to describe the polyhedral divisor we look at the projection of the cones via the cosection s. So, for instance, for the cone σ_1 we have $\sigma_1 \cap (x = 1) = \emptyset$ and $\sigma_1 \cap (x = -1) = [1, \infty)$.

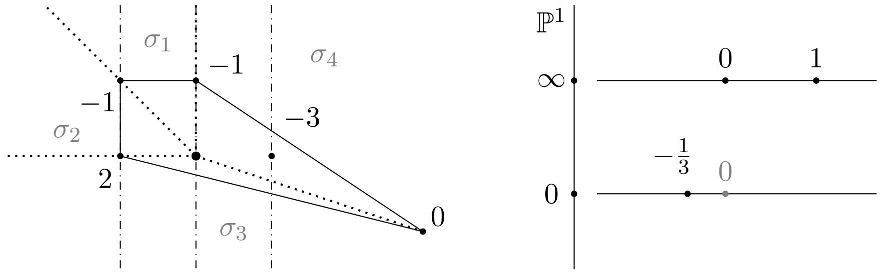


Figure 3.9: Polygon $\overline{P}$ for $\overline{X}$ and its polyhedral decompisition

Thus the associated polyhedral divisor is

$$\mathcal{D}^{\sigma_1} = \emptyset \otimes \{0\} + [1, \infty) \otimes \{\infty\} \tag{3.18}$$

and similarly for the other cones we have

$$\begin{aligned} \mathcal{D}^{\sigma_2} &= \emptyset \otimes \{0\} + [0, 1] \otimes \{\infty\} \\ \mathcal{D}^{\sigma_3} &= (-\infty, -\tfrac{1}{3}] \otimes \{0\} + (-\infty, 0] \otimes \{\infty\} \\ \mathcal{D}^{\sigma_4} &= [-\tfrac{1}{3}, \infty) \otimes \{0\} + \emptyset \otimes \{\infty\} \end{aligned} \tag{3.19}$$

Such divisors define a complete polyhedral subdivision, therefore they identify the divisorial fan $\mathcal{S} = \{\mathcal{D}^{\sigma_i}\}_{i=1..4}$.
Moreover, as explained in Lemma 3.2.1, the minimal resolution of the toric surface defined by such a divisorial fan is given by the configuration in Figure 3.8 above where the (-1)–curve identified by the vertex $(0, 1) \in \overline{P}$ and intersecting the (-3)–curve is fixed by the chosen subtorus action. Thus, the blow up at a general point on such curve is represented by the polyhedral fan in Figure 3.10 below

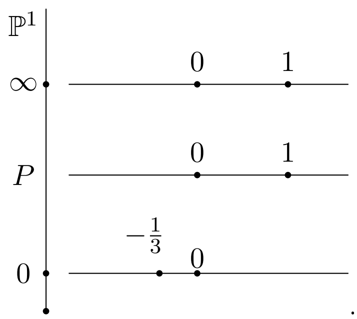


Figure 3.10: Divisorial fan for $S^3_{(0,1)}$

with associated polyhedral divisors:

$$\begin{aligned}(\mathcal{D}^{\sigma_1})' &= \emptyset \otimes \{0\} + [1,\infty) \otimes \{P\} + [1,\infty) \otimes \{\infty\} \\ (\mathcal{D}^{\sigma_2})' &= \emptyset \otimes \{0\} + [0,1] \otimes \{P\} + [0,1] \otimes \{\infty\} \\ (\mathcal{D}^{\sigma_3})' &= (-\infty,-\tfrac{1}{3}] \otimes \{0\} + (-\infty,0] \otimes \{P\} + (-\infty,0] \otimes \{\infty\} \\ (\mathcal{D}^{\sigma_4})' &= [-\tfrac{1}{3},\infty) \otimes \{0\} + \emptyset \otimes \{P\} + \emptyset \otimes \{\infty\}\end{aligned} \tag{3.20}$$

On the other hand, consider the toric surface $\mathbb{P}(1,2,5)$, with associated Fano polygon as in Figure 3.11 below, and suppose we want to mutate the edge representing the A_1 singularity. Similarly as above, if we take the subtorus action defined by $m = (1,0)$ we obtain the divisorial fan showed in the same Figure:

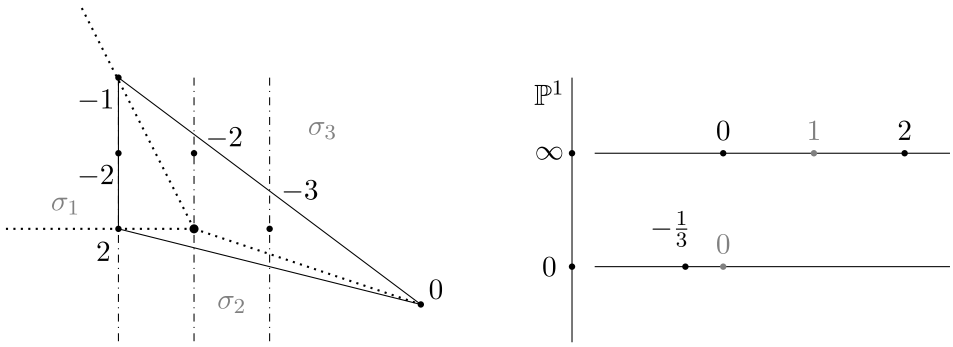


Figure 3.11: Fano polygon for $\mathbb{P}(1,2,5)$ and its divisorial fan

with associated polyhedral divisors:

$$\begin{aligned}\mathcal{D}^{\sigma_1} &= \emptyset \otimes \{0\} + [0,2] \otimes \{\infty\} \\ \mathcal{D}^{\sigma_2} &= (-\infty,-\tfrac{1}{3}] \otimes \{0\} + (-\infty,0] \otimes \{\infty\} \\ \mathcal{D}^{\sigma_3} &= [-\tfrac{1}{3},\infty) \otimes \{0\} + [2,\infty) \otimes \{\infty\}\end{aligned} \tag{3.21}$$

Hence, by considering admissible polyhedral decompositions as in 3.7, we have

$$D^{\sigma_1} = [0,1] + [0,1] \tag{3.22}$$

where $d = p = 1$, so by Remark 3.2.1 the associated deformation of the cone gives a flat family $\pi : \mathcal{X} \to \mathbb{A}^1$ where the general element is a smoothing of the A_1 singularity. Moreover, in terms of the divisorial fan, we have the admissible decomposition induced from the decomposition of the said cone, so that

$$\begin{aligned} \mathcal{S}_\infty &= \mathcal{S}^0_\infty + \mathcal{S}^1_\infty \\ \mathcal{S}_0 &= \mathcal{S}_0 + \emptyset \end{aligned} \tag{3.23}$$

where

$$\mathcal{S}^0_\infty = \mathcal{S}^1_\infty = (-\infty, 0] + [0,1] + [1,\infty)$$

Ultimately, this gives a flat family $\pi : \mathcal{X} = X(\mathcal{S}^{tot}) \to \mathbb{P}^1$ (as described in the construction of 3.2.2), where the general element has divisorial fan as in Figure 3.10, i.e. of the surface $S^3_{(0,1)}$.
As a result, the surface $S^3_{(0,1)}$ represents a qG–smoothing of the toric surface $\mathbb{P}(1,2,5)$.

Example 2: $S^{2,1}_{(2,2)} \rightsquigarrow X^{2,2}_{(2,2)}$

Similarly as example above, consider the minimal surface $S^{2,1}_{(2,2)}$ whose minimal resolution admits a configuration of curves of Type 2 that is isomorphic to the one pictured below.

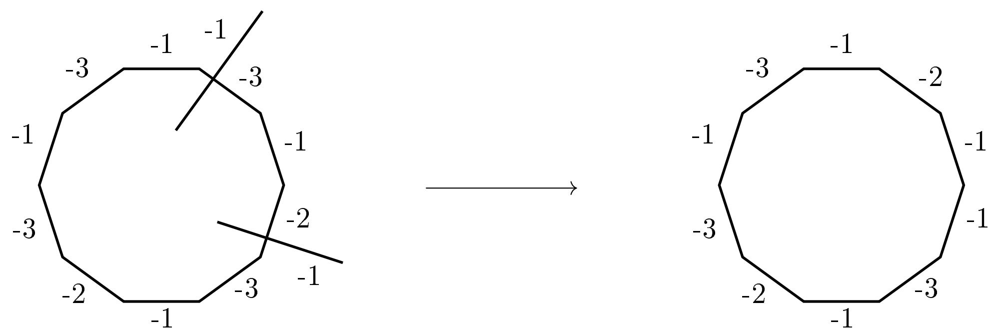


Figure 3.12: Type 2 Non toric blow up of toric curve configuration

The configuration of $S = S^{2,1}_{(2,2)}$ is obtained by blowing up a general point on a curve F_1 with self intersection (-1) and a general point on a curve F_2 with self intersection (-2) of a toric surface $\overline{X}_1$, as pictured in Figure 3.12. To construct the degeneration, consider the Fano polygon Q_1 associated to the toric surface $\overline{X}_1$: there are two subtorus actions fixing the curves from which we get the blow ups, namely the vertices of the polygon given by coordinates $v_1 = (0,-1)$ and $v_2 = (-1,0)$.

Similarly as Example 1 above, in Figure 3.13 we have the Fano polygon where the vertices are labelled as the self intersections of the curves they represent; moreover, we have fixed the subtorus action $m = (1,0)$ fixing the curve F_1, and we obtain the relative polyhedral fan. The construction of such fan is completely analogous to Example 1.

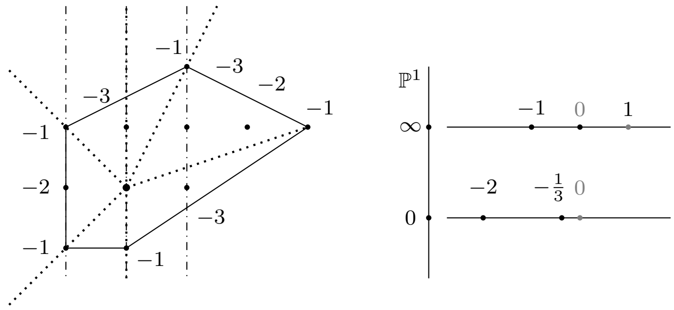


Figure 3.13: Polygon $\overline{P}_1$ for $\overline{X}_1$ and its polyhedral decompisition for $m = (1,0)$

By Theorem 3.3.2, we obtain a T–variety, say S_1, representing the general fibre of a mutation $\pi_1 : \mathcal{X}^1 \to \mathbb{P}^1$ linking the two Fano polygons pictured in Figure 3.14. In terms of integral affine variety, the surface S_1 is represented by a polygon $\overline{Q}_1 = \overline{P}_1^{\vee} \in M'_{\mathbb{Q}}$ with the addition of one focus–focus singularity.

As described in the proof of Theorem 3.3.3, by limiting the focus–focus singularity to one of the boundary points on the monodromy invariant line we obtain the degenerations of the surface S_1 to the mutation equivalent surfaces identified by the Fano polygons in Figure 3.14 below.

To obtain the configuration for the surface S we have to blow up the (-2)–curve fixed by the subtorus action $m = (0,1)$. As we can see from the Fano polygons, by blowing up the (-2)–curve from the surfaces represented by the polygon on the left in Figure 3.14 we would obtain a surface with the wrong singularity type. Thus, choose $\overline{P}_2$ to be the polygon pictured on the right and fix the subtorus action $m = (1,0)$ to contstruct the associated polyhedral fan (Figure 3.15).

Hence, once again by 3.3.1 we have a one–parameter deformation $\pi_2 : \mathcal{X}^2 \to \mathbb{P}^1$ so that the general element is given by a T–variety constructed from a Minkowski decomposition of the divisorial fan of $\overline{P}_2$. A degeneration is given by the toric surface represented by the Fano polygon in Figure 3.16.

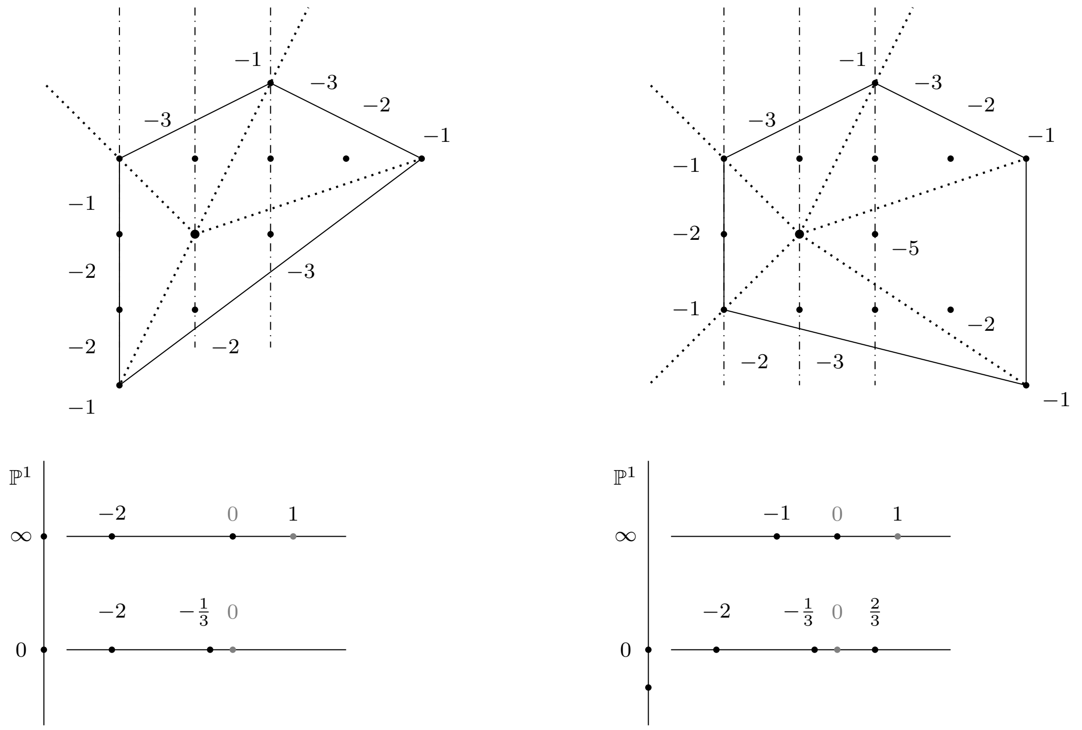


Figure 3.14: Possible polygons for $\overline{X}_2$ and their polyhedral decompisition for $m = (1, 0)$

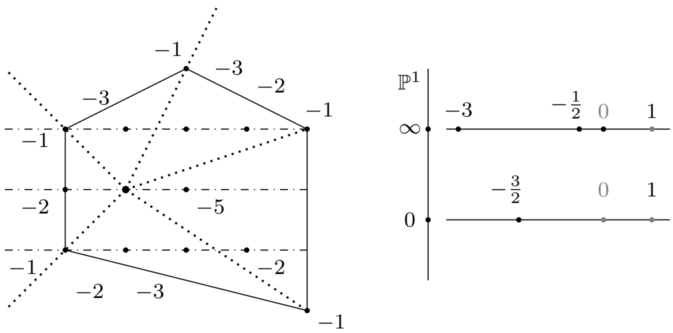


Figure 3.15: Polygon $\overline{P}_2$ for $\overline{X}_2$ and its polyhedral decompisition for $m = (0, 1)$

This degeneration can be seen on the integral affine variety represented by Q_1 with two focus–focus singularities lying on two distinct monodromy invariant lines. When both of the singularities are pushed to the boundary, we obtain the variety X identified by the polygon in Figure 3.16. This surface admits a smoothing S constructed from the two subsequent equivariant complexity 1 deformations, and represented by the integral affine variety with two focus–focus singularities.

Finally, the surface X is mutation equinvalent to the surface $X^{2,2}_{(2,2)}$ as listed in Tables

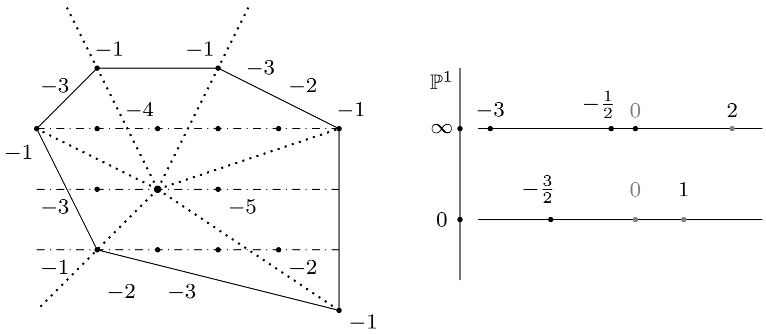


Figure 3.16: Polygon P for X and its polyhedral decompisition for $m = (0, 1)$

5.2.2, representing the minimal candidate for the mutation class. Hence, the surface S admits a toric degeneration to the surface $X^{2,2}_{(2,2)}$.

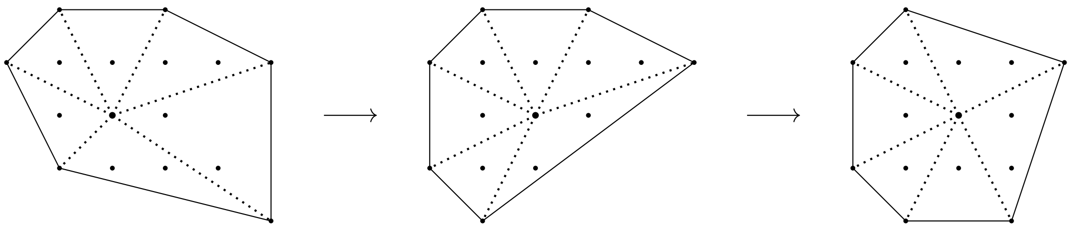


Figure 3.17: Mutations of X into $X^{2,2}_{(2,2)}$

Chapter 4

Explicit Constructions

In this Chapter we will construct the cascades from the minimal surfaces obtained from Chapter 2. We have seen in Chapter 3 how to find a toric degeneration for S by looking at the configuration of the minimal resolution of S. So firstly we will relate our minimal surfaces to the toric surfaces found from the classification of Fano polygons of given singularity content. Then we will construct the cascade of blow ups by comparing it to the toric case.

In every section we will follow these steps for the given baskets; in every cascade representation, the surfaces are divided in levels defined by n; at every level we have all the possible toric surfaces that are not mutation equivalent, thus correspond to different qG-deformation classes. The framed items represent minimal surfaces in that class and the arrows denote the blow ups/unprojections.
We will call a surface a *minimal surface* when refering to a variety found by the MMP, as listed in Table 2.5, and a *toric candidate* when refering to one of the toric surfaces that is the representative for its mutation class, as listed in Section 5.2.

4.1 Case $\mathcal{B} = \{\frac{1}{5}(1,2)\}$

4.1.1 Cascade for X with $1 \times \frac{1}{5}(1,2)$ singularities

From ([RS03]) we know that there exists a cascade of unprojections of surfaces with $1 \times \frac{1}{5}(1,2)$ singularity given by blowing up a specific head variety.

Theorem 4.1.1 ([RS03], Theorem 2.1). *Let $T = T_6 \subset \mathbb{P}(1,1,3,5)$, and for $d \leq 6$ $\sigma^{(d)} : T^{(d)} \dashrightarrow T$ the blow-up of T in d general points, E_i the exceptional curves over P_i. Then $T^{(d)}$ is a del Pezzo orbifold with singularities of type $\frac{1}{5}(1,2)$ only.*

For $d \leq 5$ there are embeddings

$$E_i \cong \mathbb{P}^1 \subset T^{(d)} \subset \mathbb{P}(1^{7-d}, 2^2, 3, 4, 5) \tag{4.1}$$

while for $T^{(6)}$ the surface embeds in $\mathbb{P}(1,2,3,4,5)$ as a complete intersection $T_{6,8}$. In particular, each map $T^{(d)} \dashrightarrow T^{(d-1)}$ is an unprojection.

In the case analysed by Reid, the head of the cascade is:

$$T = T_6 \subset \mathbb{P}(1,1,3,5)$$

For this surface we have

$$(-K_T)^2 = \frac{32}{5} \qquad \text{and} \qquad h^0(-K_T) = 7$$

which correspond to the invariants of $\mathbb{P}(1,2,5)$, thus we want to relate these two varieties by qG-deformation.

Let u, v, t be the coordinates for the weighted projective space $\mathbb{P}(1,2,5)$. Then consider the Veronese embedding of degree 2:

$$\begin{aligned} v_2 :\ \mathbb{P}(1,2,5) &\hookrightarrow \mathbb{P}(1,1,3,5) \\ (u,v,t) &\mapsto (u^2 = x, v = y, ut = z, t^2 = w) \end{aligned}$$

The image of $\mathbb{P}(1,2,5)$ is given by the sextic $xw - z^2 = 0$ in $\mathbb{P}(1,1,3,5)$. The smoothing corresponding to the surface given by the flat deformation of the sextic:

$$xw = z^2 - y^6 = (z - y^3)(z + y^3) \tag{4.2}$$

This surface has two lines $\mathbb{P}(1,5)$ meeting at the singular point $(0,0,0,1)$; as a result, the configuration of curves of this surface corresponds to the one of $S^3_{(0,1)}$. Thus $S^3_{(0,1)}$ is the qG-deformation of $\mathbb{P}(1,2,5)$ smoothing away the point $\frac{1}{2}(1,1)$, hence the surface T_6 at the head of the cascade in theorem 4.1.1.

Remark 4.1.1. The surface T_6 is not toric and its Picard rank can be calculated using the Hodge decomposition (for details, see [BF17]). Thus, $\rho(T) = h^1(T, \mathcal{O}_T^*) = h^2(T, \mathbb{Z}) = h^0(T, \mathbb{C})$ since for del Pezzo surfaces the $H^2(T, \mathbb{Z})$ has no torsion, and

$$h^2(T,\mathbb{C}) = h^{2,0} + h^{1,1} + h^{0,2} = h^{1,1}.$$

The $h^{p,q}$ are calculated from

$$h^{p,q} = h^q(\widehat{\Omega}^p_T)$$

considering $j : T^0 \hookrightarrow T$ immersion of the smooth locus of T and $\widehat{\Omega}^p_X := j_* \Omega^p_{T^0}$.
In this case, since $\omega_T = \mathcal{O}(-4)$, the if we consider the Jacobian ideal $J = (w, 6y^5, 2z, x)$ the rank of the Picard group can be read in the graded component $deg(f) - 4 = 2$ of $R = \mathbb{C}[x,y,z,w]/(J)$ so that $\rho(T) = |R(2)| + 1$. As a result, $R(2) = \langle y \rangle$, thus $\rho(T) = 2$.

Similarly, we can check for the weighted projective space $\mathbb{P}(1,3,5)$ that via the Veronese map of degree 3

$$\begin{aligned} v_3 :\ & \mathbb{P}(1,3,5) \hookrightarrow \mathbb{P}(1,1,2,5) \\ & (u,v,t) \mapsto (u^3 = x, v = y, ut = z, t^3 = w) \end{aligned}$$

it embeds as a sextic $(xw = z^3) \subset \mathbb{P}(1,1,2,5)$. Smoothing away from the $\frac{1}{3}(1,2) = A_2$ singularity, gives the non–toric sextic

$$xw = z^3 - y^6 = (z - y^2)(z - \zeta y^2)(z - \zeta^2 y^2) \tag{4.3}$$

(where ζ is a primitive 3-rd root of unity) with 3 rational lines passign through the point $(0,0,0,1)$. So, as above, the minimal surface $S^4_{(0,1)}$ represent the smoothing of the toric $\mathbb{P}(1,3,5)$ and they are in the same qG-deformation class.

Now, consider the toric surfaces occurring from the classification up to mutation in Chapter 3, we have a (concurring) chain of blow ups as represented in Figure 4.1 below.
The listed toric surfaces represent all the degenerations available for the respective classes defined by the invariants. The minimal ones (framed in the picture) are the base of the cascade and they are the degenerations of (respectively) $S^3_{(0,1)}$ and $S^4_{(0,1)}$, as we discussed above. Moreover, the configurations of their minimal resolutiona are of Type 1, thus we can apply Theorem 3.3.2 to explicitly find a toric degeneration (as explained, for instance, in Example 1 of Section 3.4).

Theorem 4.1.2. *There are 9 qG-deformation classes of del Pezzo surfaces with* $1 \times \frac{1}{5}(1,2)$ *orbifold points:*

- *if* $K_S^2 = \frac{32}{5}$*, then* $S = T_6 \subset \mathbb{P}(1,1,3,5)$ *and it is minimal;*

- *if* $K_S^2 = \frac{27}{5}$, *then either* $S = T^{(1)} \subset \mathbb{P}(1^6, 2^2, 3, 4, 5)$ *or* $S = V_6 \subset \mathbb{P}(1, 1, 2, 5)$, *where* V_6 *is minimal;*

- *if* $K_S^2 = \frac{22}{5}$, *then either* $S = T^{(2)} \subset \mathbb{P}(1^5, 2^2, 3, 4, 5)$ *or* $S = V^{(1)}$*;*

- *if* $\frac{2}{5} < K_S^2 \leq \frac{17}{5}$, *then* $d = 3, 4, 5$ *and* $S = T^{(d)} = V^{(d-1)} \subset \mathbb{P}(1^{7-d}, 2^2, 3, 4, 5)$*;*

- *if* $K_S^2 = \frac{2}{5}$, *then* $S = T^{(6)} = V^{(5)} = T_{6,8} \subset \mathbb{P}(1, 2, 3, 4, 5)$*;*

where $T^{(d)} = Bl_{P_1,\ldots,P_d}(T_6)$ *and* $V^{(d-1)} = Bl_{Q_1,\ldots,Q_{d-1}}(V_6)$ *represent blow–ups of the minimal surface at respectively d or d-1 general points.*

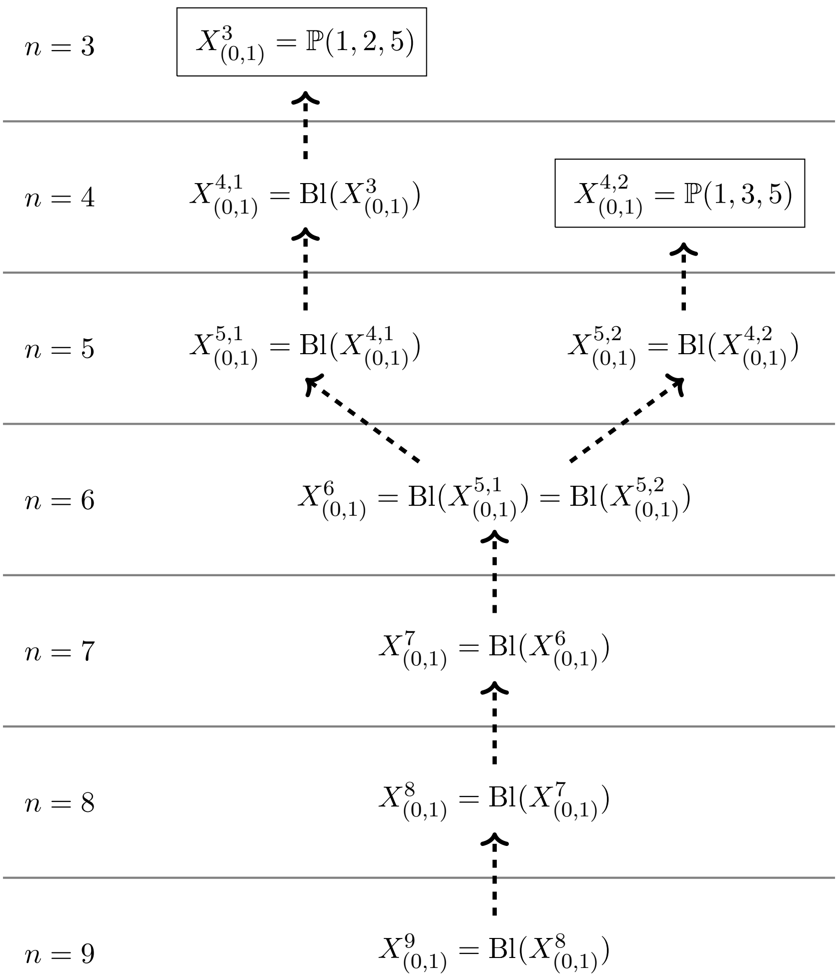


Figure 4.1: Cascades for surfaces of type (0,1)

4.2 Case $\mathcal{B} = \{k_1 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2)\}$

4.2.1 (1,1)

Unlike the case $(0,1)$ it was not as straightforward to find the degenerations of the minimal surfaces just by working on the structure of their graded rings. Nevertheless, we can recover their degenerations from the curve configurations of their minimal resolution. $S^2_{(1,1)}$ is toric, thus the general element of the family is isomorphic to $X^2_{(1,1)}$, and this is the only possible surface with $\rho = 2$ and degree $K^2 = \frac{86}{15}$. At level $n = 3$ we have four qG–deformation classes, one of which is represented by the blow up of the surface $S^2_{(1,1)}$ and admits a degeneration to $X^{3,4}_{(1,1)}$. The surfaces $S^{3,2}_{(1,1)}$ and $S^{3,3}_{(1,1)}$ have configurations of Type 1 and we can again use the methods in 3.3.1 to recover the degeneration. Thus we are left with the surface $S^{3,1}_{(1,1)}$ which must correspond to the smoothing of the toric candidate $X^{3,1}_{(1,1)}$. Alternatively, we can find an isomorphism to a surface with a Type 2 configuration and compute the degeneration to the said toric surface.
Finally, at level $n = 4$ we have three toric candidates, two of them appear as blow ups of surfaces at level $n = 3$ (as indicated in Figure 4.2 below), while the surface $X^{4,2}_{(1,1)}$ is minimal and must admit a qG–smoothing; therefore, the surface $S^4_{(1,1)}$ represents the general element of the family. Moreover, such surface admits a configuration of Type 2, thus we can again use Theorem 3.3.3 to explicitly find the degeneration. There is only one toric surface at level $n = 7$ and it represents blow ups of both surfaces $X^{6,1}_{(1,1)}, X^{6,2}_{(1,1)}$. Thus there is a unique (toric) degeneration class at level $n = 7$.

As in the previous section, the framed surfaces represent the toric degenerations of the minimal surfaces; so, to sum up the degenerations are:

$$S^2_{(1,1)} = X^2_{(1,1)}$$

$$S^{3,1}_{(1,1)} \rightsquigarrow X^{3,2}_{(1,1)}$$
$$S^{3,2}_{(1,1)} \rightsquigarrow X^{3,1}_{(1,1)}$$
$$S^{3,3}_{(1,1)} \rightsquigarrow X^{3,3}_{(1,1)}$$
$$S^{4}_{(1,1)} \rightsquigarrow X^{4,2}_{(1,1)}$$

Theorem 4.2.1. *There are 13 qG–deformation classes of del Pezzo surfaces with $\frac{1}{3}(1,1)+\frac{1}{5}(1,2)$ orbifold points, and they all admit a toric degeneration. A surface S of this kind is one of the following:*

- *If $K_S^2 = \frac{86}{15}$ then $S = S^2_{(1,1)} = T$, and T is minimal.*
- *If $K_S^2 = \frac{71}{15}$ then either $S = T^{(1)}$ or $S = S^{3,1}_{(1,1)} = U$ or $S^{3,2}_{(1,1)} = V$ or $S^{3,3}_{(1,1)} = W$,*

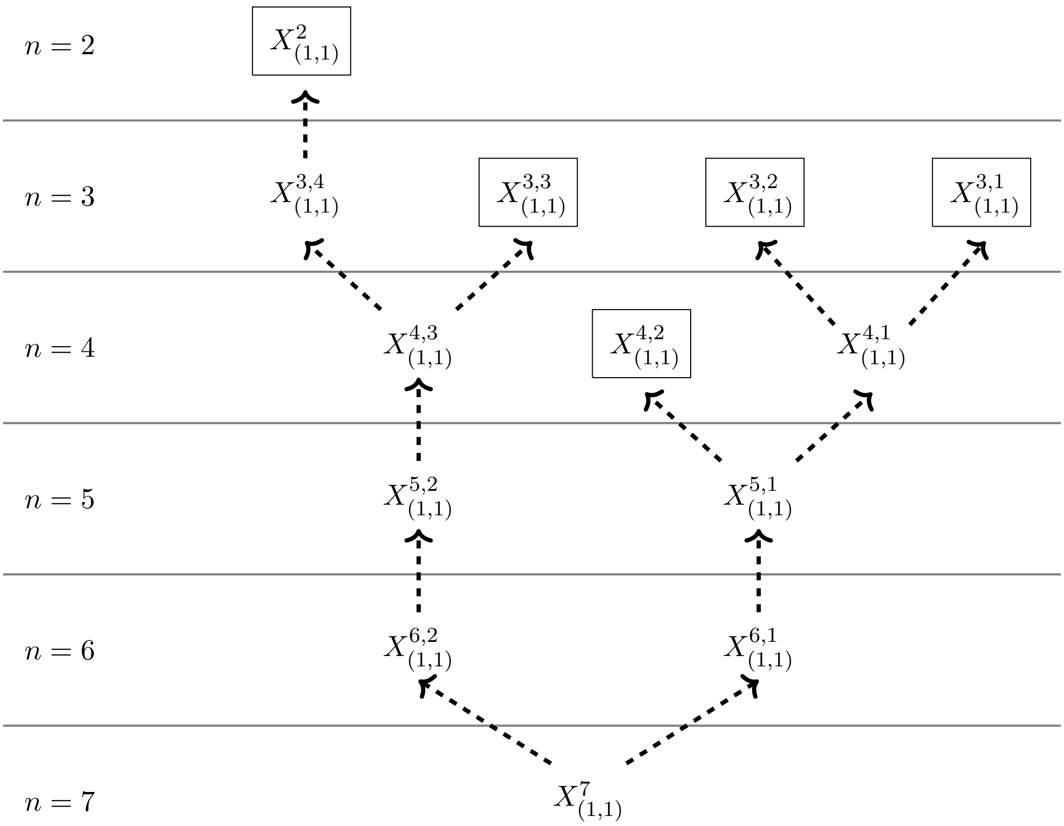


Figure 4.2: Cascades for surfaces of type (1,1)

where U, V and W are minimal.

- *If $K_S^2 = \frac{56}{15}$ then either $S = T^{(2)} = W^{(1)}$ or $S = U^{(1)} = V^{(1)}$ or $S = S^4_{(1,1)} = Z$, where Z is minimal.*
- *If $K_S^2 = \frac{41}{15}$ then either $S = T^{(3)} = W^{(2)}$ or $S = U^{(2)} = V^{(2)} = Z^{(1)}$.*
- *If $K_S^2 = \frac{26}{15}$ then either $S = T^{(4)} = W^{(3)}$ or $S = U^{(3)} = V^{(3)} = Z^{(2)}$.*
- *If $K_S^2 = \frac{11}{15}$ then $S = T^{(6)} = W^{(4)} = U^{(4)} = V^{(4)} = Z^{(3)}$.*

where $T^{(i)}$, $U^{(i)}$, $V^{(i)}$, $W^{(i)}$ and $Z^{(i)}$ represent blow ups of respectively T, U, V, W and Z in i general points.

4.2.2 (2,1)

In this case, the two minimal surfaces at level $n = 1$ are toric, thus the bases of the two main cascades are $X^{1,1}_{(2,1)}, X^{1,2}_{(2,1)}$, and the surfaces at levels $n = 2, 3$ are blow ups of the two respective surfaces. At level $n = 4$ we have the blow ups from level $n = 3$ and a new minimal surface $S^4_{(1,1)}$ which represents the smoothing of the toric surface $X^4_{(1,1)}$. Thus, for the minimal surfaces we have:

$$
\begin{aligned}
S^{1,1}_{(2,1)} &= X^{1,1}_{(2,1)} \\
S^{1,2}_{(2,1)} &= X^{1,2}_{(2,1)} \\
S^{4}_{(2,1)} &\rightsquigarrow X^{4,3}_{(2,1)}
\end{aligned}
$$

At level $n = 5$ there is one toric candidate only, and it arises as the blow up of each surface at level $n = 4$. Thus there is only one qG–deformation class for the relative invariants.

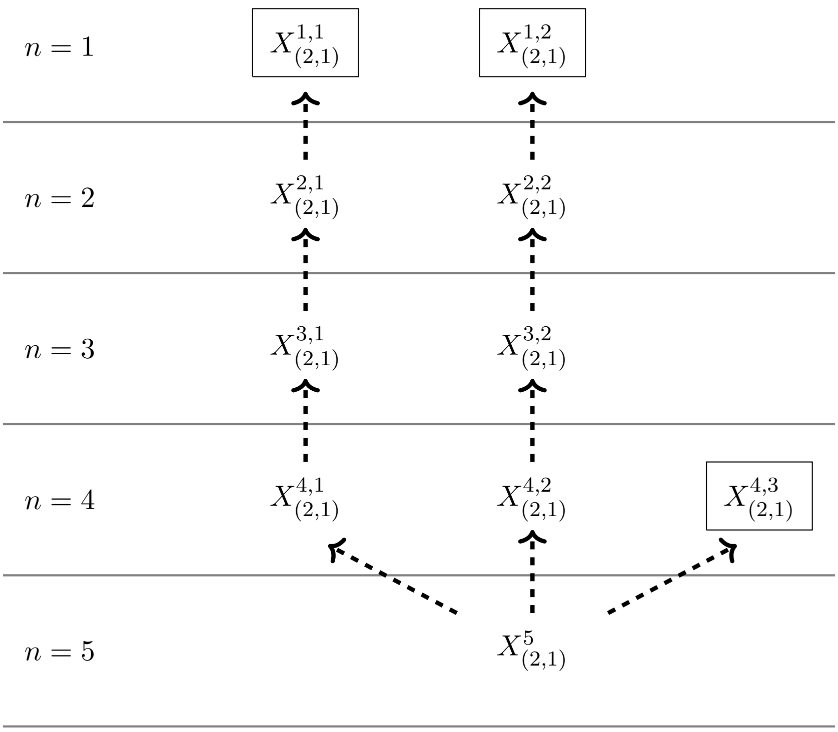


Figure 4.3: Cascades for surfaces of type (2,1)

Theorem 4.2.2. *There are 10 qG–deformation classes of del Pezzo surfaces with $2 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2)$ orbifold points admitting a toric degeneration. A surface S of this kind is one of the following:*

- *If $K_S^2 = \frac{76}{15}$ then either $S = S^{1,1}_{(2,1)} = T$ or $S = S^{1,2}_{(2,1)} = U$, where T and U are minimal;*
- *If $K_S^2 = \frac{61}{15}$ then either $S = T^{(1)}$ or $S = U^{(1)}$;*

- *If* $K_S^2 = \frac{46}{15}$ *then either* $S = T^{(2)}$ *or* $S = U^{(2)}$*;*
- *If* $K_S^2 = \frac{31}{15}$ *then either* $S = T^{(3)}$ *or* $S = U^{(3)}$ *or* $S = S^4_{(2,1)} = V$*, where* V *is minimal.*
- *If* $K_S^2 = \frac{16}{15}$ *then* $S = T^{(4)} = U^{(4)} = V^{(1)}$*.*

where $T^{(i)}$*,* $U^{(i)}$*,*$V^{(i)}$ *represent blow ups of respectively* T, U, V *in* i *general points.*

4.2.3 (3,1)

There is just one minimal surface with $h^0(-K) \neq 0$ and of type $(3, 1)$ and has a configuration of Type 1. Thus we can check that

$$S^2_{(3,1)} \rightsquigarrow X^2_{(3,1)}$$

As there are no other candidates, for this case we have only one cascade and the picture is as follows:

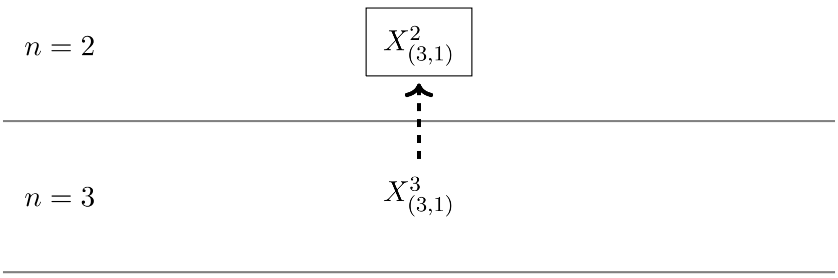


Figure 4.4: Cascades for surfaces of type (3,1)

Theorem 4.2.3. *There are 2 qG–deformation classes of del Pezzo surfaces with* $3 \times \frac{1}{3}(1, 1) + \frac{1}{5}(1, 2)$ *orbifold points admitting a toric degeneration:*

- *If* $K_S^2 = \frac{12}{5}$ *then* $S = S^{2,1}_{(3,1)} = T$ *and it is minimal;*
- *If* $K_S^2 = \frac{7}{5}$ *then* $S = T^{(1)}$*, blow up of* T *in one general point.*

4.2.4 (4,1)

In this case we have only one toric candidate $X^1_{(4,1)}$ and one isomorphism class. Thus we have:

$$S^1_{(4,1)} \rightsquigarrow X^1_{(4,1)}$$

Theorem 4.2.4. *There is only 1 qG–deformation class of del Pezzo surfaces with* $4 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2)$ *admitting a toric degeneration: the general element of the family is* $S^{1,1}_{(4,1)}$ *and has* $K_S^2 = \frac{26}{15}$ *and* $n = 1$.

4.3 Case $\mathcal{B} = \{k_1 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2)\}$

4.3.1 (0,2)

This case is analogous to the case $(2,1)$: the surfaces at the base of the cascades are toric and we can determine members of the cascades until level $n = 6$. At level $n = 5$, we have a new minimal surface with a Type 1 configuration, so we can easily recover its degeneration to the toric candidate $X^{5,3}_{(0,2)}$.

$$S^{2,1}_{(0,2)} = X^{2,1}_{(0,2)}$$
$$S^{2,2}_{(0,2)} = X^{2,2}_{(0,2)}$$
$$S^{5}_{(0,2)} \rightsquigarrow X^{5,3}_{(0,2)}$$

At level $n = 6$ we have two nonminimal toric candidates for the qG–class, but at this stage we do not know of which surface they are the blow up of.

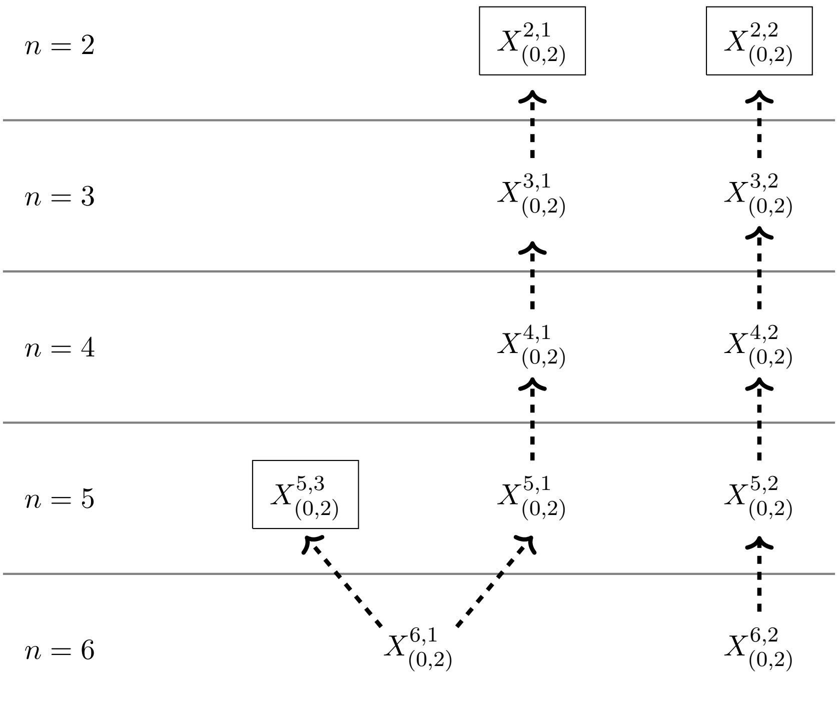


Figure 4.5: Cascades for surfaces of type (0,2)

Theorem 4.3.1. *There are 11 qG–deformation classes of del Pezzo surfaces with $2 \times \frac{1}{5}(1,2)$ orbifold points and they all admit a toric degeneration:*

- *If* $K_S^2 = \frac{24}{5}$ *then either* $S = S^{2,1}_{(0,2)} = T$ *or* $S = S^{2,2}_{(0,2)} = U$*, where* T *and* U *are minimal;*
- *If* $K_S^2 = \frac{19}{5}$ *then either* $S = T^{(1)}$ *or* $S = U^{(1)}$*;*
- *If* $K_S^2 = \frac{14}{5}$ *then either* $S = T^{(2)}$ *or* $S = U^{(2)}$*;*
- *If* $K_S^2 = \frac{9}{5}$ *then either* $S = T^{(3)}$ *or* $S = U^{(3)}$ *or* $S = S^{5}_{(0,2)} = V$*, where* V *is minimal.*
- *If* $K_S^2 = \frac{4}{5}$ *then either* $S = T^{(4)} = V^{(1)}$ *or* $S = U^{(4)}$*.*

where $T^{(i)}, U^{(i)}, V^{(i)}$*, represent blow ups of respectively* T, U, V *in* i *general points.*

4.3.2 (1,2)

There is only one surface at level $n = 2$, namely $S^2_{(1,2)}$, and its configuration is of Type 1. At level $n = 3$ we have four distinct toric candidates and three minimal surfaces all admitting a Type 1 configuration. As one of the toric candidates is the blow up of the toric surface at level $n = 2$, then we have the following degenerations:

$$\begin{aligned}
S^2_{(1,2)} &\rightsquigarrow X^2_{(1,2)} \\
S^{3,1}_{(1,2)} &\rightsquigarrow X^{3,2}_{(1,2)} \\
S^{3,2}_{(1,2)} &\rightsquigarrow X^{3,4}_{(1,2)} \\
S^{3,3}_{(1,2)} &\rightsquigarrow X^{3,3}_{(1,2)}
\end{aligned}$$

Hence, by reconstructing the blow ups of the toric surfaces we have:

Theorem 4.3.2. *There are 8 qG–deformation classes of del Pezzo surfaces with* $\frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2)$ *orbifold points admitting a toric degeneration. If* $K^2 \neq \frac{17}{15}$

- *If* $K_S^2 = \frac{47}{15}$ *then* $S = S^2_{(1,2)} = T$ *where* T *is minimal;*
- *If* $K_S^2 = \frac{32}{15}$ *then either* $S = T^{(1)}$ *or* $S = S^{3,1}_{(1,2)} = U$ *or* $S = S^{3,2}_{(1,2)} = V$ *or* $S = S^{3,3}_{(1,2)} = W$*, where* U, V, W *are minimal;*
- *if* $K_S^2 = \frac{17}{15}$*, then either* $S = T^{(2)}$ *or* $S = U^{(1)}$ *or* $S = V^{(1)} = W^{(1)}$

where $T^{(1)}$*, represents the blow up of* T *in a general point.*

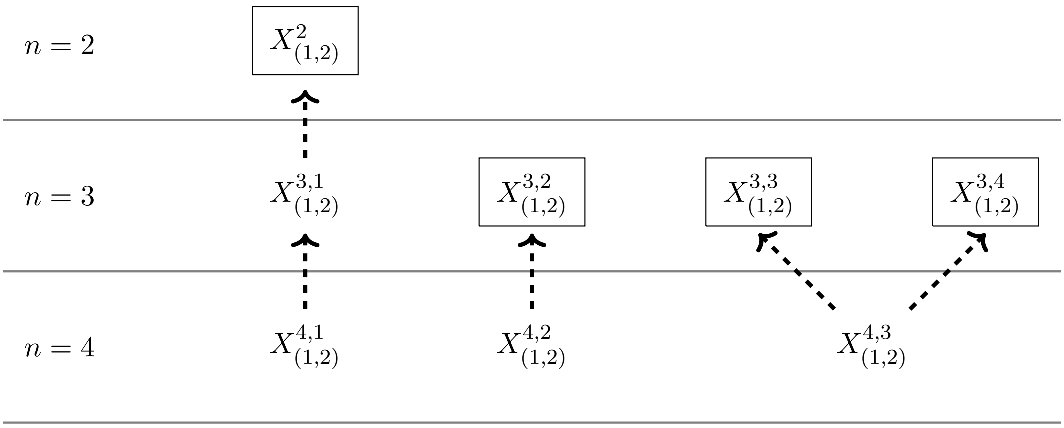


Figure 4.6: Cascades for surfaces of type (1,2)

4.3.3 (2,2)

The minimal surface $S^{1}_{(2,2)}$ at level $n = 1$ is toric and it is the only available surface with such invariants. At level $n = 2$ we have four distinct isomorphism classes of minimal surfaces, but they all admit a Type 1 or Type 2 configuration. Thus we find the following degenerations:

$$
\begin{aligned}
S^{1}_{(2,2)} &\rightsquigarrow X^{1}_{(2,2)} \\
S^{2,1}_{(2,2)} &\rightsquigarrow X^{2,2}_{(2,2)} \\
S^{2,2}_{(2,2)} &\rightsquigarrow X^{2,4}_{(2,2)} \\
S^{2,3}_{(2,2)} &\rightsquigarrow X^{2,3}_{(2,2)} \\
S^{2,4}_{(2,2)} &\rightsquigarrow X^{2,5}_{(2,2)}
\end{aligned}
$$

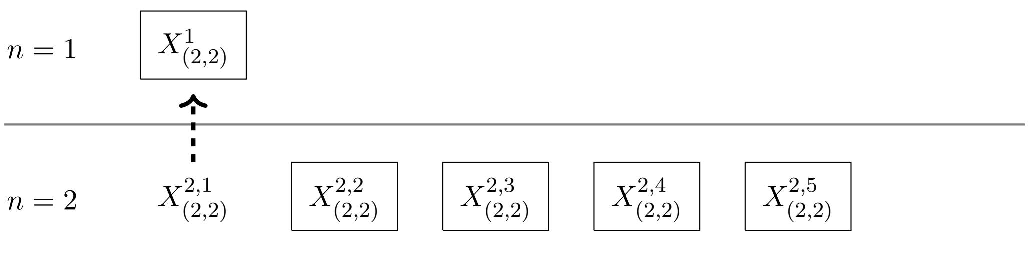


Figure 4.7: Cascades for surfaces of type (2,2)

Theorem 4.3.3. *There are 6 qG–deformation classes of del Pezzo surfaces with* $2 \times$

$\frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2)$ *orbifold points admitting a toric degeneration:*

- *If* $K_S^2 = \frac{37}{15}$ *then* $S = S^1_{(2,2)} = T$ *where* T *is minimal;*
- *If* $K_S^2 = \frac{22}{15}$ *then either* $S = T^{(1)}$ *or* $S = S^{2,2}_{(2,2)} = U$ *or* $S = S^{2,3}_{(2,2)}V$ *or* $S = S^{2,4}_{(2,2)}W$*, where* U, V, W *are minimal;*

where $T^{(1)}$*, represents the blow up of* T *in a general point.*

4.3.4 (3,2)

In this case the two minimal surfaces are toric and coincide with the ony two possible candidates of toric surfaces of type $(3,2)$. In particular, these surfaces have $n = 0$, therefore they are qG–rigid and they are the only surfaces in their class.

$$S^{0,1}_{(3,2)} = X^{0,1}_{(3,2)}$$
$$S^{0,2}_{(3,2)} = X^{0,2}_{(3,2)}$$

$n = 0$	$\boxed{X^{0,1}_{(3,2)}}$	$\boxed{X^{0,2}_{(3,2)}}$

Figure 4.8: Cascades for surfaces of type (3,2)

Theorem 4.3.4. *There are 2 qG–deformation classes of del Pezzo surfaces with* $3 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2)$ *orbifold points admitting a toric degeneration, i.e.* $S^{0,1}_{(3,2)}$ *and* $S^{0,2}_{(3,2)}$*. In particular these surfaces are toric.*

4.4 Case $\mathcal{B} = \{k_1 \times \frac{1}{3}(1,1) + 3 \times \frac{1}{5}(1,2)\}$

4.4.1 (0,3)

For this case we have two minimal surfaces at level $n = 3$. They all have configurations of Type 2, thus it is possible to find degenerations to the two toric candidates. Eventually, for this singularity type we will have only two qG–classes with $h^0(-K) \neq 0$ having $K^2 = \frac{6}{5}$.

$$S^{3,1}_{(0,3)} \rightsquigarrow X^{3,1}_{(0,3)}$$
$$S^{3,2}_{(0,3)} \rightsquigarrow X^{3,2}_{(0,3)}$$

$$n = 3 \qquad \boxed{X^{3,1}_{(0,3)}} \qquad \boxed{X^{3,2}_{(0,3)}}$$

Figure 4.9: Cascades for surfaces of type (0,3)

Theorem 4.4.1. *There are 2 qG–deformation classes of del Pezzo surfaces with* $3 \times \frac{1}{5}(1,2)$ *orbifold points admitting a toric degeneration, i.e.* $S^{3,1}_{(0,3)}$ *and* $S^{3,2}_{(0,3)}$.

4.4.2 (1,3)

The minimal surfaces coming from birational constructions are three distinct isomorphism classes and three toric candidates. All of the minimal surfaces have Type 1 configurations, so we can easily recover the degenerations:

$$S^{1,1}_{(1,3)} \rightsquigarrow X^{1,3}_{(1,3)}$$
$$S^{1,2}_{(1,3)} \rightsquigarrow X^{1,1}_{(1,3)}$$
$$S^{1,3}_{(1,3)} \rightsquigarrow X^{1,2}_{(1,3)}$$

$n = 1$ $\boxed{X^{1,1}_{(1,3)}}$ $\boxed{X^{1,2}_{(1,3)}}$ $\boxed{X^{1,2}_{(1,3)}}$

Figure 4.10: Cascades for surfaces of type (1,3)

Theorem 4.4.2. *There are 3 qG–deformation classes of del Pezzo surfaces with* $1 \times \frac{1}{3}(1,1) + 3 \times \frac{1}{5}(1,2)$ *orbifold points admitting a toric degeneration, i.e.* $S^{1,1}_{(1,3)}$, $S^{1,2}_{(1,3)}$ *and* $S^{1,3}_{(1,3)}$ *having* $K^2 = \frac{23}{15}$.

4.5 Case $\mathcal{B} = \{4 \times \frac{1}{5}(1,2)\}$

In this case the only isomorphism classes for minimal surfaces of such singularity type correspond with the two available toric surfaces at level $n = 0$. So, similarly to case (3,2), they represent the only surface in their family.

$$S^{0,1}_{(0,4)} = X^{0,1}_{(0,4)}$$
$$S^{0,2}_{(0,4)} = X^{0,2}_{(0,4)}$$

$$n = 0 \qquad \boxed{X^{0,1}_{(0,4)}} \qquad \boxed{X^{0,2}_{(0,4)}}$$

Figure 4.11: Cascades for surfaces of type (0,4)

Theorem 4.5.1. *There are 2 qG–deformation classes of del Pezzo surfaces with $4 \times \frac{1}{5}(1,2)$ orbifold points admitting a toric degeneration, i.e. $S^{0,1}_{(0,4)}$ and $S^{0,2}_{(0,4)}$. In particular these surfaces are toric.*

4.6 Conclusions and Further work

The cascade constructions listed in the previous section allow us to find all of the possible deformation classes for surfaces admitting toric degenerations. So, by counting such classes we get

Theorem 4.6.1. *Let S be a del Pezzo surface with singularity content*

$$\left(n, k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2)\right).$$

with $k_2 \geq 1$. Then there are 69 qG–classes of such surfaces admitting a toric degeneration.

We have mentioned that in [ACHK15] it has been conjectured that there is a one–to–one correspondence between

$$\begin{pmatrix} \text{Fano polygons} \\ \text{up to mutation} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{qG-deformation classes of locally qG-rigid} \\ \text{del Pezzo surfaces of class TG} \\ \text{w/ cyclic quotient singularities} \end{pmatrix}$$

From our classification we find the conjecture holds for our specific case, where we have assumed that $h^0(-K) \neq 0$. Nevertheless some of the cascade constructions are not complete, as we could have surfaces with $h^0(-K) = 0$. It would be interesting to see if our methods to construct qG–deformations as described in Chapter 3 can be generalised to this type of surfaces. Thus the next step would be to see how many of the blow ups of the surfaces at the bottom of each cascade represent distinct qG–deformation classes in the case of surfaces not admitting a toric boundary.
Ultimatley, we would be able to find all of the possible qG–deformation classes for surfaces with singularity content $(n, k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2))$.

The methods we used for the birational constructions seem to leave room for an extensive generalisation. The case–by–case analysis necessary to find the possible models lead us to interesting combinatorial problems that deal with both analysis of invariants and curve configurations. Indeed, the Directed Minimal Model Program we have worked with relies on very explicit analysis of such curves, and they strictly depend on the singularity type.
This also influences the geometric invariants (e.g. degree or Picard rank), and have mutual dependence with the configuration of curves. It would therefore be interesting to find an organic way to apply these methods to other classes of rigid singularities and understand a broader class of said surfaces.

Despite having a neat description in terms of cascades, in our case the graded rings have rather complicated structures which make calculating explicit biregular model particularly arduous.
As we have mentioned, in the case of [CH15] we have a complete biregular classification for the orbifold del Pezzo surfaces with $\frac{1}{3}(1,1)$ by means of specific degeneracy loci. This method is strictly related to Laurent inversion ([CKP17]), and represents a (possible) good alternative to overcome computational difficulties given by the graded ring structure.

To conclude, the methods we have used can be useful to build an algorithmic way of approaching the problem of classifying del Pezzo orbifolds with any given type of singularity.

Chapter 5

Tables

5.1 Isomorphism types for Minimal Surfaces

Surface	Contractions	S^{min}	$\rho(X)$	n	K_S^2
$S^3_{(0,1)}$	$\mathcal{C}5$	$\mathbb{P}(1,1,2)$	2	3	$\frac{32}{5}$
$S^4_{(0,1)}$	$\mathcal{C}8+\mathcal{C}2$	$\mathbb{P}^2$	3	4	$\frac{27}{5}$
$S^2_{(1,1)}$	$\mathcal{C}6$	$\mathbb{P}(1,2,5)$	2	2	$\frac{86}{15}$
$S^{3,1}_{(1,1)}$	$\mathcal{C}5+\mathcal{C}1$	$\mathbb{P}(1,1,3)$	3	3	$\frac{71}{15}$
$S^{3,2}_{(1,1)}$	$\mathcal{C}6+\mathcal{C}8$	$\mathbb{P}(1,2,3)$	3	3	$\frac{71}{15}$
$S^{3,3}_{(1,1)}$	$\mathcal{C}8+\mathcal{C}2$	$\mathbb{P}(1,1,3)$	3	3	$\frac{71}{15}$
$S^4_{(1,1)}$	$\mathcal{C}6+\mathcal{C}1$	$\mathbb{P}(1,1,2)$	4	4	$\frac{56}{15}$
$S^{1,1}_{(2,1)}$	$\mathcal{C}10$	$\mathbb{P}(1,1,3)$	2	1	$\frac{76}{15}$
$S^{1,2}_{(2,1)}$	$\mathcal{C}11$	$\mathbb{P}(1,3,5)$	2	1	$\frac{76}{15}$
$S^4_{(2,1)}$	$\mathcal{C}6+\mathcal{C}1$	$\mathbb{P}(1,1,3)$	5	5	$\frac{31}{15}$
$S^2_{(3,1)}$	$\mathcal{C}6+\mathcal{C}1+\mathcal{C}10$	$\mathbb{P}(1,1,3)$	4	2	$\frac{12}{5}$
$S^1_{(4,1)}$	$\mathcal{C}6+\mathcal{C}7+\mathcal{C}10$	$\mathbb{P}(1,1,3)$	4	1	$\frac{47}{15}$
$S^{2,1}_{(0,2)}$	$\mathcal{C}5$	$\mathbb{P}(1,2,5)$	2	2	$\frac{24}{5}$
$S^{2,2}_{(0,2)}$	$\mathcal{C}8$	$\mathbb{P}(1,3,5)$	2	2	$\frac{24}{5}$

Surface	Contractions	S^{min}	$\rho(X)$	n	K_S^2
$S^5_{(0,2)}$	$\mathcal{C}8+\mathcal{C}2+\mathcal{C}8+\mathcal{C}2$	$\mathbb{P}^2$	5	5	$\frac{9}{5}$
$S^2_{(1,2)}$	$\mathcal{C}5+\mathcal{C}9$	$\mathbb{P}(1,1,3)$	3	2	$\frac{62}{15}$
$S^{3,1}_{(1,2)}$	$\mathcal{C}5+\mathcal{C}5+\mathcal{C}9$	$\mathbb{P}(1,1,2)$	4	3	$\frac{47}{15}$
$S^{3,2}_{(1,2)}$	$\mathcal{C}5+\mathcal{C}6+\mathcal{C}1$	$\mathbb{P}(1,2,5)$	4	3	$\frac{47}{15}$
$S^{3,3}_{(1,2)}$	$\mathcal{C}6+\mathcal{C}1+\mathcal{C}8$	$\mathbb{P}(1,3,5)$	4	3	$\frac{47}{15}$
$S^1_{(2,2)}$	$\mathcal{C}6+\mathcal{C}10$	$\mathbb{P}(1,1,2)$	3	1	$\frac{37}{15}$
$S^{2,1}_{(2,2)}$	$\mathcal{C}5+\mathcal{C}1+\mathcal{C}10$	$\mathbb{P}(1,1,3)$	4	2	$\frac{22}{15}$
$S^{2,2}_{(2,2)}$	$\mathcal{C}5+\mathcal{C}1+\mathcal{C}11$	$\mathbb{P}(1,3,5)$	4	2	$\frac{22}{15}$
$S^{2,3}_{(2,2)}$	$\mathcal{C}8+\mathcal{C}2+\mathcal{C}10$	$\mathbb{P}(1,1,3)$	4	2	$\frac{22}{15}$
$S^{2,4}_{(2,2)}$	$\mathcal{C}8+\mathcal{C}2+\mathcal{C}11$	$\mathbb{P}(1,3,5)$	4	2	$\frac{22}{15}$
$S^{0,1}_{(3,2)}$	$\mathcal{C}10+\mathcal{C}10$	$\mathbb{P}(1,1,3)$	3	0	$\frac{9}{5}$
$S^{0,2}_{(3,2)}$	$\mathcal{C}10+\mathcal{C}11$	$\mathbb{P}(1,3,5)$	3	0	$\frac{9}{5}$
$S^{3,1}_{(0,3)}$	$\mathcal{C}5+\mathcal{C}1+\mathcal{C}5$	$\mathbb{P}(1,2,5)$	4	3	$\frac{6}{5}$
$S^{3,2}_{(0,3)}$	$\mathcal{C}5+\mathcal{C}1+\mathcal{C}8$	$\mathbb{P}(1,3,5)$	4	3	$\frac{6}{5}$
$S^{1,1}_{(1,3)}$	$\mathcal{C}5+\mathcal{C}10$	$\mathbb{P}(1,2,5)$	3	1	$\frac{23}{15}$
$S^{1,2}_{(1,3)}$	$\mathcal{C}8+\mathcal{C}10$	$\mathbb{P}(1,3,5)$	3	1	$\frac{23}{15}$
$S^{1,3}_{(1,3)}$	$\mathcal{C}8+\mathcal{C}12$	$\mathbb{P}(3,4,5)$	3	1	$\frac{23}{15}$
$S^{0,1}_{(0,4)}$	$\mathcal{C}13$	R	2	0	$\frac{8}{5}$
$S^{0,2}_{(0,4)}$	$2\times\mathcal{F}3$		2	0	$\frac{8}{5}$

Table 5.1: Minimal surfaces of type (k_1, k_2) and extremal contractions

5.2 Toric surfaces with singularity content $(n, k_1 \times \frac{1}{3}(1,1) + k_2 \times \frac{1}{5}(1,2))$

$(0,1)$			
Surface	Vertices	n	K_X^2
$X^3_{(0,1)}$	$(1,5),(2,5),(-1,-3)$	3	$\frac{32}{5}$
$X^{4,1}_{(0,1)}$	$(3,5),(4,5),(-2,-3)$	4	$\frac{27}{5}$
$X^{4,2}_{(0,1)}$	$(1,5),(2,5),(-1,-4)$	4	$\frac{27}{5}$
$X^{5,1}_{(0,1)}$	$(1,5),(2,5),(-1,-3),(0,-1)$	5	$\frac{22}{5}$
$X^{5,2}_{(0,1)}$	$(1,5),(2,5),(-1,-4),(0,1)$	5	$\frac{22}{5}$
$X^6_{(0,1)}$	$(1,5),(2,5),(-1,-4),(-1,-3)$	6	$\frac{17}{5}$
$X^7_{(0,1)}$	$(1,5),(2,5),(-2,-7)$	7	$\frac{12}{5}$
$X^8_{(0,1)}$	$(1,3),(1,-4),(-2,3)$	8	$\frac{7}{5}$
$X^9_{(0,1)}$	$(1,2),(11,-3),(-13,2)$	9	$\frac{2}{5}$

Table 5.2: Mutation classes of toric surfaces with singularity content $(n, \frac{1}{5}(1,2))$

(1,1)			
Surface	Vertices	n	K_X^2
$X^2_{(1,1)}$	$(1,5),(2,5),(-1,-3),(-1,-2)$	2	$\frac{86}{15}$
$X^{3,1}_{(1,1)}$	$(1,3),(2,3),(2,1),(1,0),(-2,-1)$	3	$\frac{71}{15}$
$X^{3,2}_{(1,1)}$	$(1,3),(2,3),(2,1),(-1,-1),(-2,-1)$	3	$\frac{71}{15}$
$X^{3,3}_{(1,1)}$	$(1,5),(2,5),(2,1),(0,-1),(-1,-2)$	3	$\frac{71}{15}$
$X^{3,4}_{(1,1)}$	$(3,5),(4,5),(-1,-2),(-2,-3),(1,2)$	3	$\frac{71}{15}$
$X^{4,1}_{(1,1)}$	$(1,3),(2,3),(2,1),(1,0),(-1,-1),(-2,-1)$	4	$\frac{56}{15}$
$X^{4,2}_{(1,1)}$	$(1,5),(2,5),(1,1),(1,0),(-1,-3)$	4	$\frac{56}{15}$
$X^{4,3}_{(1,1)}$	$(1,5),(2,5),(2,1),(0,-1),(-1,-3),(-1,-2)$	4	$\frac{56}{15}$
$X^{5,1}_{(1,1)}$	$(1,3),(2,3),(2,1),(0,-1),(-2,-1)$	5	$\frac{41}{15}$
$X^{5,2}_{(1,1)}$	$(1,5),(2,5),(-1,-4),(-1,-2)$	5	$\frac{41}{15}$
$X^{6,1}_{(1,1)}$	$(1,2),(1,-2),(-3,1),(-3,2)$	6	$\frac{26}{15}$
$X^{6,2}_{(1,1)}$	$(1,2),(2,1),(-1,-3),(-1,2)$	6	$\frac{26}{15}$
$X^{7}_{(1,1)}$	$(1,2),(5,-2),(-7,1),(-9,2)$	7	$\frac{11}{15}$

Table 5.3: Mutation classes of toric surfaces with singularity content $(n, \frac{1}{3}(1,1)+\frac{1}{5}(1,2))$

(2, 1)			
Surface	Vertices	n	K_X^2
$X_{(2,1)}^{1,1}$	$(1,5),(2,5),(1,1),(-1,-2)$	1	$\frac{76}{15}$
$X_{(2,1)}^{1,2}$	$(1,3),(2,3),(1,0),(-2,-1)$	1	$\frac{76}{15}$
$X_{(2,1)}^{2,1}$	$(1,5),(2,5),(1,1),(0,-1),(-1,-2)$	2	$\frac{61}{15}$
$X_{(2,1)}^{2,2}$	$(1,5),(2,5),(1,1),(-1,-4)(0,1)$	2	$\frac{61}{15}$
$X_{(2,1)}^{3,1}$	$(1,3),(2,3),(1,0),(0,-1),(-2,-1)$	3	$\frac{46}{15}$
$X_{(2,1)}^{3,2}$	$(1,5),(2,5),(1,1),(-1,-3),(-1,-2)$	3	$\frac{46}{15}$
$X_{(2,1)}^{4,1}$	$(1,2),(2,1),(-1,-2),(-2,-1),(-1,2)$	4	$\frac{31}{15}$
$X_{(2,1)}^{4,2}$	$(1,2),(2,1),(-1,-2),(-2,1),(-1,2)$	4	$\frac{31}{15}$
$X_{(2,1)}^{4,3}$	$(1,3),(2,3),(1,0),(-3,-4)$	4	$\frac{31}{15}$
$X_{(2,1)}^{5}$	$(1,2),(2,1),(1,-2),(-7,2)$	5	$\frac{16}{15}$

Table 5.4: Mutation classes of toric surfaces with singularity content $(n, 2 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2))$

(3, 1)			
Surface	Vertices	n	K_X^2
$X_{(3,1)}^{2,1}$	$(1,3),(2,3),(1,0),(-1,-2),(-2,-1)$	2	$\frac{12}{5}$
$X_{(3,1)}^{2,2}$	$(3,5),(4,5),(-1,-2),(-3,-4),(0,1)$	2	$\frac{12}{5}$
$X_{(3,1)}^{3}$	$(1,2),(3,1),(0,-1),(-3,-2),(-3,2)$	3	$\frac{7}{5}$

Table 5.5: Mutation classes of toric surfaces with singularity content $(n, 3 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2))$

(4, 1)			
Surface	Vertices	n	K_X^2
$X_{(4,1)}^{1}$	$(1,2),(2,1),(1,-1),(-1,-2),(-2,-1),(-1,2)$	1	$\frac{47}{15}$

Table 5.6: Mutation classes of toric surfaces with singularity content $(n, 4 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2))$

$(0, 2)$			
Surface	Vertices	n	K_X^2
$X_{(0,2)}^{2,1}$	$(3, 5), (4, 5), (1, 1), (-4, -5)$	2	$\frac{24}{5}$
$X_{(0,2)}^{2,2}$	$(1, 5), (2, 5), (1, 0), (-1, -1)$	2	$\frac{24}{5}$
$X_{(0,2)}^{3,1}$	$(3, 5), (4, 5), (-3, -5), (-2, -3), (1, 2)$	3	$\frac{19}{5}$
$X_{(0,2)}^{3,2}$	$(1, 5), (2, 5), (1, 0), (0, -1), (-1, -1)$	3	$\frac{19}{5}$
$X_{(0,2)}^{4,1}$	$(1, 5), (2, 5), (0, -1), (-2, -5)$	4	$\frac{14}{5}$
$X_{(0,2)}^{4,2}$	$(1, 5), (2, 5), (1, 0), (-1, -2), (-1, -1)$	4	$\frac{14}{5}$
$X_{(0,2)}^{5,1}$	$(1, 5), (2, 5), (2, 3), (-2, -5)$	5	$\frac{9}{5}$
$X_{(0,2)}^{5,2}$	$(1, 5), (2, 5), (2, 3), (1, -1), (-2, -1)$	5	$\frac{9}{5}$
$X_{(0,2)}^{5,3}$	$(1, 5), (2, 5), (-3, -10)$	5	$\frac{9}{5}$
$X_{(0,2)}^{6,1}$	$(1, 2), (4, -1), (3, -2), (-6, 1), (-7, 2)$	6	$\frac{4}{5}$
$X_{(0,2)}^{6,2}$	$(1, 2), (4, -1), (3, -2), (-1, -1), (-7, 2)$	6	$\frac{4}{5}$

Table 5.7: Mutation classes of toric surfaces with singularity content $(n, 2 \times \frac{1}{5}(1, 2))$

$(1, 2)$			
Surface	Vertices	n	K_X^2
$X_{(1,2)}^{2}$	$(1, 5), (2, 5), (1, 0), (-1, -2)$	2	$\frac{62}{15}$
$X_{(1,2)}^{3,1}$	$(1, 5), (2, 5), (1, 0), (-1, -3), (-1, -1)$	3	$\frac{47}{15}$
$X_{(1,2)}^{3,2}$	$(1, 3), (2, 3), (-1, -1), (-2, -1)$	3	$\frac{47}{15}$
$X_{(1,2)}^{3,3}$	$(1, 5), (2, 5), (1, 1), (-1, -5)$	3	$\frac{47}{15}$
$X_{(1,2)}^{3,4}$	$(1, 5), (2, 5), (-1, -5), (-1, -2)$	3	$\frac{47}{15}$
$X_{(1,2)}^{4,1}$	$(1, 3), (2, 3), (2, -1), (-1, -2), (-3, -1)$	4	$\frac{32}{15}$
$X_{(1,2)}^{4,2}$	$(1, 2), (2, 1), (1, -2), (-5, -1), (-5, -2)$	4	$\frac{32}{15}$
$X_{(1,2)}^{4,3}$	$(1, 2), (3, 1), (-1, -2), (-5, -2)$	4	$\frac{32}{15}$

Table 5.8: Mutation classes of toric surfaces with singularity content $(n, \frac{1}{3}(1, 1) + 2 \times \frac{1}{5}(1, 2))$

$(2,2)$			
Surface	Vertices	n	K_X^2
$X^1_{(2,2)}$	$(1,5),(2,5),(1,0),(-1,-3),(-1,-2)$	1	$\frac{37}{15}$
$X^{2,1}_{(2,2)}$	$(1,2),(2,1),(1,-2),(-1,-2),(-2,-1),(-1,2)$	2	$\frac{12}{15}$
$X^{2,2}_{(2,2)}$	$(1,2),(2,1),(1,-2),(-2,-1),(-2,-1),(-1,2)$	2	$\frac{12}{15}$
$X^{2,3}_{(2,2)}$	$(1,2),(2,1),(1,-2),(-3,1),(-3,2)$	2	$\frac{12}{15}$
$X^{2,4}_{(2,2)}$	$(1,3),(2,3),(2,1),(-1,-3),(-2,-3),(-1,1)$	2	$\frac{12}{15}$
$X^{2,5}_{(2,2)}$	$(1,2),(2,-1),(1,-3),(-1,-1),(-2,1),(-1,2)$	2	$\frac{12}{15}$

Table 5.9: Mutation classes of toric surfaces with singularity content $(n, 2\times\frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2))$

$(3,2)$			
Surface	Vertices	n	K_X^2
$X^{0,1}_{(3,2)}$	$(1,3),(2,3),(1,0),(-1,-3),(-2,-1)$	0	$\frac{9}{5}$
$X^{0,2}_{(3,2)}$	$(1,3),(2,3),(1,-1),(-1,-2),(-2,-1)$	0	$\frac{9}{5}$

Table 5.10: Mutation classes of toric surfaces with singularity content $(n, 3\times\frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2))$

$(0,3)$			
Surface	Vertices	n	K_X^2
$X^{3,1}_{(0,3)}$	$(1,2),(3,1),(-1,-2),(-4,1),(-3,2)$	3	$\frac{6}{5}$
$X^{3,2}_{(0,3)}$	$(1,2),(3,1),(-1,-2),(-3,-1),(-3,2)$	3	$\frac{6}{5}$

Table 5.11: Mutation classes of toric surfaces with singularity content $(n, 3\times\frac{1}{5}(1,2))$

$(1,3)$			
Surface	Vertices	n	K_X^2
$X^{1,1}_{(1,3)}$	$(1,2),(2,1),(1,-2),(-2,-1),(-1,2)$	1	$\frac{23}{15}$
$X^{1,2}_{(1,3)}$	$(1,2),(3,1),(-1,-2),(-2,1),(-1,2)$	1	$\frac{23}{15}$
$X^{1,3}_{(1,3)}$	$(1,2),(2,-1),(1,-3),(-2,1),(-1,2)$	1	$\frac{23}{15}$

Table 5.12: Mutation classes of toric surfaces with singularity content $(n, \frac{1}{3}(1,1) + 3 \times \frac{1}{5}(1,2))$

$(0,4)$			
Surface	Vertices	n	K_X^2
$X^{0,1}_{(0,4)}$	$(1,5),(2,5),(1,0),(-2,-5)$	0	$\frac{8}{5}$
$X^{0,2}_{(0,4)}$	$(1,5),(2,5),(-1,-5),(-2,-5)$	0	$\frac{8}{5}$

Table 5.13: Mutation classes of toric surfaces with singularity content $(n, 4 \times \frac{1}{5}(1,2))$

Appendix:
Case analysis of the Minimal Model Program trees

In this appendix we report the case analysis described by the trees of possibilities for the directed minimal model programs introduced in Section 3.3.

Analysis of case $(2,1)$

Case 1 $(\mathcal{C}5) + (\mathcal{C}1):\ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow 2 \times \frac{1}{3} \qquad S_{(2,0)}$

As in case 1 of case analysis for $(1,1)$, we check amongst minimal surfaces of type $(2,0)$: from the classification in [CH15] we have 2 cases of minimal surfaces with $2 \times \frac{1}{3}(1,1)$ singularities:

(A) The first minimal surface of type $(2,0)$ is a hypersurface $X_6 \subset \mathbb{P}(1,1,3,3)$, which has $\rho = 6$, so not our case;

(B) From blow ups of the other surface $S^3_{(2,0)}$ we obtain a configuration with a floating (-1)-curve.

Case 2 $(\mathcal{C}5) + (\mathcal{C}6) + (\mathcal{C}1) + (\mathcal{C}6) + (\mathcal{C}1):\ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

This configuration also will contain a floating (-1)-curve, giving a non minimal surface.

Case 3 $(\mathcal{C}5) + (\mathcal{C}6) + (\mathcal{C}1) + (\mathcal{C}7):\ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow \text{smooth} \qquad (\mathbb{P}^2 \text{ or } \mathbb{P}^1 \times \mathbb{P}^1)$

In both cases we obtain a non minimal surface, as starting from $\mathbb{P}^2$ we have a floating (-1)-curve and from $\mathbb{P}^1 \times \mathbb{P}^1$ we have a non directed MMP.

Case 4 $(\mathcal{C}5) + (\mathcal{C}6) + (\mathcal{C}7):\ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

This case also gives a configuration with a floating (-1)-curve, thus is non minimal.

Case 5 $(\mathcal{C}5) + (\mathcal{C}7): \ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow \frac{1}{3} \qquad S_{(1,0)}$

For the surface $S^5_{(1,0)}$ we would have a $(\mathcal{C}6)$ contraction available before $(\mathcal{C}7)$, so we will consider only the case of $\mathbb{P}(1,1,3)$. The resulting configuration turns out to be non minimal because of a floating (-1) curve.

Case 6 (A) $(\mathcal{C}5) + (\mathcal{C}11) + (\mathcal{C}2): \ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

The configuration admits a floating (-1)-curve.

(B) $(\mathcal{C}5) + (\mathcal{C}11): \ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 + A_2 \qquad \mathbb{P}(1,2,3)$

Similarly, this configuration gives a floating (-1)-curve.

Case 7 $(\mathcal{C}6) + (\mathcal{C}1): \ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow \frac{1}{3} + \frac{1}{5} \qquad S_{(1,1)}$

We end up with a surface of type $(1,1)$, so we have 4 possible minimal surfaces to start from:

(A) $S^4_{(1,1)}$: we obtain a non directed MMP as a contraction of type $(\mathcal{C}5)$ is available;

(B) $S^{3,1}_{(1,1)}$: as above, a $(\mathcal{C}5)$ is available;

(C) $S^{3,2}_{(1,1)}$: also here, a $(\mathcal{C}5)$ is available;

(D) $S^{3,3}_{(1,1)}$: blowing up this surface, we obtain a non toric configuration for a surface $S^4_{(2.1)}$ with following invariants:

$$K^2 = \frac{31}{15} \qquad h^0(-K) = 2 \qquad \rho = 5 \qquad n = 4$$

A similar configuration coming from the blow up of a different (-2)-curve to obtain $\frac{1}{5}(1,2)$ point gives a non directed MMP with a $(\mathcal{C}5)$ available.

Case 8 (A) $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}1) + (\mathcal{C}8) + (\mathcal{C}2): \ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

From this case we have a non minimal surface with a floating (-1)-curve, so non minimal.

(B) $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}1) + (\mathcal{C}8): \ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 + A_2 \qquad \mathbb{P}(1,2,3)$

Similarly, we obtain a floating (-1)-curve in the configuration.

(C) $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}1): \ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 + \frac{1}{5} \qquad \mathbb{P}(1,2,5)$

Also in this case we obtain a floating (-1)-curve in the configuration.

Case 9 $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}1) + (\mathcal{C}9) 2 \times \frac{1}{3} + \frac{1}{5}: \longrightarrow \text{smooth} \qquad (\mathbb{P}^2 \text{ or } \mathbb{P}^1 \times \mathbb{P}^1)$

In both cases we have a non minimal surface, obtaining a floating (-1)-curve $(\mathbb{P}^2)$ or a non directed MMP $(\mathbb{P}^1 \times \mathbb{P}^1)$;

Case 10 $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}9):\ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

After first blow ups we obtain a contraction of type $(\mathcal{C}1)$ is available, giving a non directed MMP.

Case 11 $(\mathcal{C}6) + (\mathcal{C}7):\ 2 \times \frac{1}{3} + \frac{1}{5} : \longrightarrow \frac{1}{5} \qquad S_{(0,1)}$

Ending up with a surface of type $(0,1)$, we have 2 cases to consider:

(A) $S^3_{(0,1)}$: this case gives a non directed MMP as a $(\mathcal{C}5)$ would be available;

(B) $S^4_{(0,1)}$: again this is not directed because of a contraction of type $(\mathcal{C}1)$ available.

Case 12 $(\mathcal{C}6) + (\mathcal{C}9):\ 2 \times \frac{1}{3} + \frac{1}{5} : \longrightarrow \frac{1}{3} \qquad S_{(1,0)}$

Again we consider only the surface $\mathbb{P}(1,1,3)$ of type $(1,0)$ as $S(1,0)^5$ would have a $(\mathcal{C}6)$ contraction available before $(\mathcal{C}9)$. After the first blow up we have a floating (-1)-curve, thus the resulting surface is non minimal.

Case 13 $(\mathcal{C}6) + (\mathcal{C}10) 2 \times \dfrac{1}{3} + \dfrac{1}{5} \xrightarrow{(\mathcal{C}6)} A_1 + \dfrac{1}{3} + \dfrac{1}{5} \xrightarrow{(\mathcal{C}10)} A_1$

Similarly to te case above, we obtain a floating (-1)-curve in the configuration.

Case 14 $(\mathcal{C}6) + (\mathcal{C}12) + (\mathcal{C}3):\ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

From this sequance of blow ups we obtain 2 different configurations of curves, but for both of them the MMP is not directed as contractions of type $(\mathcal{C}5)$ or $(\mathcal{C}6)$ are available before $(\mathcal{C}12)$.

Case 15 $(\mathcal{C}8) + (\mathcal{C}2):\ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow 2 \times \frac{1}{3} \qquad S_{(2,0)}$

As in case 1, we consider blow ups of the surface $S^3_{(2,0)}$, but these give a surface where a contraction of type $(\mathcal{C}6)$ is available contradicting minimality of contractions.

Case 16 $(\mathcal{C}10):\ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow \frac{1}{3} \qquad \mathbb{P}(1,1,3)$

The blow ups of the minimal surface $\mathbb{P}(1,1,3)$ give the toric configuration with invariants:

$$K^2 = \frac{76}{15} \qquad h^0(-K) = 5 \qquad \rho = 2 \qquad n = 1$$

Case 17 (A) $(\mathcal{C}11) + (\mathcal{C}2) \qquad (\mathbb{P}^2 \text{ or } \mathbb{P}^1 \times \mathbb{P}^1)):\ 2 \times \frac{1}{3} + \frac{1}{5} \longrightarrow \frac{1}{5} \qquad S_{(0,1)}$

From surfaces of type $(0,1)$ we would have contractions of type $(\mathcal{C}6)$ or $(\mathcal{C}8)$ available, so the MMP would not be directed;

(B) $(\mathcal{C}11):\ 2\times\frac{1}{3}+\frac{1}{5}\longrightarrow A_2+\frac{1}{5}\qquad \mathbb{P}(1,3,5)$

This sequence gives a toric configuration similar to case (16):

$$K^2=\frac{76}{15}\qquad h^0(-K)=5\qquad \rho=2\qquad n=1$$

Case 18 (A) $(\mathcal{C}12)+(\mathcal{C}3):\ 2\times\frac{1}{3}+\frac{1}{5}\longrightarrow\frac{1}{3}\qquad \mathbb{P}(1,1,3)$

From both cases of type $(1,0)$ we obtain configurations with floating (-1)-curves;

(B) $(\mathcal{C}12):\ 2\times\frac{1}{3}+\frac{1}{5}\longrightarrow A_3+\frac{1}{3}\qquad \mathbb{P}(1,3,4)$

For blow ups from $\mathbb{P}(1,3,4)$ we obtain same configurations as cases (16) and (17), making this a non directed sequence.

As a result, the minimal models for a surface with $2\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)$ points are

(I) $S^{1,1}_{(2,1)}$ and $S^{1,2}_{(2,1)}$ with $\rho=2$ and $K^2=\frac{76}{15}$

(II) $S^{4}_{(2,1)}$ with $\rho=5$ and $K^2=\frac{31}{15}$

Analysis for case (3,1)

Case 1 $(\mathcal{C}5)+(\mathcal{C}1):\ 3\times\frac{1}{3}+\frac{1}{5}\longrightarrow 3\times\frac{1}{3}(1,1)\qquad S_{(3,0)}$

We look again at the classification from [CH15] to find surfaces of type $(3,0)$ with $\rho\le 3$: there exists one surface of this kind with $\rho=3$, i.e. $S^2_{(3,0)}$ so by taking the given blow ups we obtain a nontoric configuration with a floating curve.

Case 2 $(\mathcal{C}5)+(\mathcal{C}6)+(\mathcal{C}1)+(\mathcal{C}7):\ 3\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)\longrightarrow\frac{1}{3}(1,1)$

From the listed blow ups of $\mathbb{P}(1,1,3)$ we obtain a non toric configuration where a contraction of type $(\mathcal{C}1)$ is available before $(\mathcal{C}6)$, so the MMP is not directed.

Case 3 $(\mathcal{C}5)+(\mathcal{C}6)+(\mathcal{C}1)+(\mathcal{C}11):\ 3\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)\longrightarrow A_1+A_2\qquad \mathbb{P}(1,2,3)$

This MMP is not directed as after the first blow up we have a contraction of type $(\mathcal{C}7)$ available before $(\mathcal{C}11)$.

Case 4 $(\mathcal{C}5)+(\mathcal{C}6)+(\mathcal{C}7)+(\mathcal{C}7):\ 3\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)\longrightarrow\mathbb{P}^2$

Blowing up from the smooth model with $\rho=1$ we obtain a non directed MMP as other contractions are available.

Case 5 $(\mathcal{C}5) + (\mathcal{C}7): \ 3 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{3}(1,1) \qquad S_{(2,0)}$

There is a unique minimal surface of type $(2,0)$ with $\rho = 3$, namely $S^4_{(2,0)}$, but we obtain a configuration with a floating (-1)-curve.

Case 6 $(\mathcal{C}6) + (\mathcal{C}1): \ 3 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \qquad S_{(2,1)}$

We have 2 distinct minimal surfaces of type $(2,1)$, both with $\rho = 2$. From applying the same blow ups to both surfaces we obtain two isomorphic models with invariants

$$K^2 = \frac{12}{5} \qquad h^0(-K) = 2 \qquad \rho = 4 \qquad n = 2$$

giving a cunfiguration for the surface $S^2_{(3,1)}$

Case 7 $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}1) + (\mathcal{C}9): \longrightarrow \frac{1}{3}(1,1) \qquad \mathbb{P}(1,1,3)$

From the blow up of the minimal surface $\mathbb{P}(1,1,3)$ we get a non directed MMP as a contraction of type $(\mathcal{C}1)$ before the second $(\mathcal{C}6)$ contraction.

Case 8 $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}1) + (\mathcal{C}10): \ 3 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

After the blow up of $\mathbb{P}(1,1,2)$ a contraction of type $(\mathcal{C}9)$ results available before $(\mathcal{C}10)$, so the MMP is not directed.

Case 9 (A) $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}7) + (\mathcal{C}8): \ 3 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \longrightarrow A_1 + A_2 \qquad \mathbb{P}(1,2,3)$

The resulting configurations have a floating (-1)-curve, so they cannot be minimal.

(B) $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}7): \ 3 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \longrightarrow A_1 + \frac{1}{5}(1,2) \qquad \mathbb{P}(1,2,5)$

The blow ups give the same configurations as in case (6), so the MMP is not directed.

Case 10 $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}7) + (\mathcal{C}9): \ 3 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \longrightarrow \text{smooth} \qquad (\mathbb{P}^2 \text{ or } \mathbb{P}^1 \times \mathbb{P}^1)$

Case 11 $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}8): \ 3 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \longrightarrow 2 \times A_1 + A_2 + \frac{1}{3}(1,1) \qquad \mathcal{F}1 + \mathcal{F}2$

The configuration admits a contraction of type $(\mathcal{C}7)$, so the MMP is not directed.

Case 12 $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}12): \ 3 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \longrightarrow 2 \times A_1 + A_3 \qquad M_{2,2} \text{ or } \mathcal{F}0 + \mathcal{F}1$

In both cases we obtain a contraction of type $(\mathcal{C}1)$ is available, so the surfaces are non minimal.

Case 13 $(\mathcal{C}6) + (\mathcal{C}7): \ 3 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \longrightarrow \frac{1}{3}(1,1) + \frac{1}{5}(1,2)$

We have one surface of type $(1,1)$ with $\rho = 2$ and three with $\rho = 3$ we can blow up from:

(A) $S^2_{(1,1)}$: a contraction of type $(\mathcal{C}6)$ would be available before $(\mathcal{C}7)$

(B) $S^{3,1}_{(1,1)}$: a contraction of type $(\mathcal{C}5)$ would be available

(C) $S^{3,2}_{(1,1)}$: a contraction of type $(\mathcal{C}6)$ would be available before $(\mathcal{C}7)$

(D) $S^{3,3}_{(1,1)}$: a contraction of type $(\mathcal{C}6)$ would be available before $(\mathcal{C}7)$

so none of the candidates give a minimal surface.

Case 14 $(\mathcal{C}6)+(\mathcal{C}8)+(\mathcal{C}2)+(\mathcal{C}7):\ 3\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)\longrightarrow\frac{1}{3}(1,1)\qquad\mathbb{P}(1,1,3)$

The sequence gives a non toric configuration for a surface that is isomorphic to the configuration obtained in Case (1).

Case 15 $(\mathcal{C}8)+(\mathcal{C}2):\ 3\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)\longrightarrow 3\times\frac{1}{3}(1,1)$

Similarly to case (1), by taking blow ups of the surface $S^2_{(3,0)}$, we obtain a contraction of type $(\mathcal{C}5)$ available at the beginning, so the surface is non minimal.

Case 16 $(\mathcal{C}10):\ 3\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)\longrightarrow 2\times\frac{1}{3}(1,1)$

By blowing up the minimal surface $S^4_{(2,0)}$ we end up with a non minimal configuration with a $(\mathcal{C}6)$ contraction available, so the MMP is not directed.

Case 17 $(\mathcal{C}12)+(\mathcal{C}3):\ 3\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)\longrightarrow 2\times\frac{1}{3}(1,1)$

Analogously to the situation above in case (15), there is a contraction of type $(\mathcal{C}6)$ available before $(\mathcal{C}12)$, so the surface is non minimal.

As a result, the minimal model for a surface with $3\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)$ points is $S^2_{(3,1)}$ with $\rho=4$ and $K^2=\frac{12}{5}$.

Analysis for case (4,1)

Case 1 $(\mathcal{C}6)+(\mathcal{C}7):\ 4\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)\longrightarrow 2\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)\qquad S_{(2,1)}$

From both surfaces of type $(2,1)$ and $\rho=2$ we get two configurations, namely:

(A) from $S^{1,1}_{(2,1)}$ obtain the surface $S^{1,1}_{(4,1)}$ with invariants

$$K^2=\frac{47}{15}\qquad h^0(-K)=1\qquad \rho=4\qquad n=1$$

(B) from $S^{1,2}_{(2,1)}$ we get the same configuration as $S^{1,1}_{(4,1)}$.

Case 2 $(\mathcal{C}6)+(\mathcal{C}10)+(\mathcal{C}11):\ 4\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)\longrightarrow A_1+A_2\qquad\mathbb{P}(1,2,3)$

The MMP is not directed as there exists a $(\mathcal{C}7)$ contraction available before the $(\mathcal{C}10)$.

Case 3 (A) $(\mathcal{C}8)+(\mathcal{C}11)+(\mathcal{C}11):\ 4\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)\longrightarrow 3\times A_2$

The sequence gives a configuration that is isomorphic to Case (1).

(B) $(\mathcal{C}8)+(\mathcal{C}11)\longrightarrow 2\times\mathcal{F}2$

The sequence gives the same configuration as Case (A) above, so the MMP is not directed.

Thus we end up with the minimal surface $S^{1,1}_{(4,1)}$ with $\rho=4$ and $K^2=\dfrac{47}{15}$

Analysis for case (0,2)

Case 1 We have two endpoints, one with the two possible minimal surfaces of type $(0,1)$ and $\mathbb{P}(1,2,5)$:

(A) $(\mathcal{C}5)+(\mathcal{C}1):\ 2\times\frac{1}{5}(1,2)\longrightarrow\frac{1}{5}(1,2)\qquad S^3_{(0,1)}$

We obtain a non toric configuration for a surface with a floating (-1)–curve, which is thus not minimal.

(B) $(\mathcal{C}5)+(\mathcal{C}1):\ 2\times\frac{1}{5}(1,2)\longrightarrow\frac{1}{5}(1,2)\qquad S^4_{(0,1)}$

The configuration has a floating (-1)-curve, it is not minimal

(C) $(\mathcal{C}5):\ 2\times\frac{1}{5}(1,2)\longrightarrow A_1+\frac{1}{5}(1,2)\qquad\mathbb{P}(1,2,5)$

The sequence gives the toric configuration of the surface $S^{2,1}_{(0,2)}$ with invariants:

$$K^2=\frac{24}{5}\qquad h^0(-K)=5\qquad\rho=2\qquad n=2$$

Case 2 $(\mathcal{C}5)+(\mathcal{C}5)+(\mathcal{C}1):\ 2\times\frac{1}{5}(1,2)\longrightarrow A_1\qquad\mathbb{P}(1,1,2)$

From this sequence we obtain the same configuration as case (1), so the MMP is not directed.

Case 3 (A) $(\mathcal{C}5)+(\mathcal{C}8)+(\mathcal{C}2):\ 2\times\frac{1}{5}(1,2)\longrightarrow A_1\qquad\mathbb{P}(1,1,2)$

Again, we obtain the same configuration as in case (1).

(B) $(\mathcal{C}5)+(\mathcal{C}8):\ 2\times\frac{1}{5}(1,2)\longrightarrow A_1+A_2$ The configuration will contain a floating (-1)-curve

Case 4 $(\mathcal{C}5) + (\mathcal{C}9) :\ 2 \times \frac{1}{5}(1,2) \longrightarrow \text{smooth} \qquad \mathbb{P}^2 \text{ or } \mathbb{P}^1 \times \mathbb{P}^1$

In both cases we get configurations with $(\mathcal{C}8)$ contraction available before $(\mathcal{C}9)$.

Case 5 Ending with a surface of type $(0,1)$ or $\mathbb{P}(1,3,5)$, we get:

(A) $(\mathcal{C}8) + (\mathcal{C}2) :\ 2 \times \frac{1}{5}(1,2) \longrightarrow \frac{1}{5}(1,2) \qquad S^3_{(0,1)}$

We have a floating (-1)-curve, so the surface is not minimal.

(B) $(\mathcal{C}8) + (\mathcal{C}2) :\ 2 \times \frac{1}{5}(1,2) \longrightarrow \frac{1}{5}(1,2) \qquad S^4_{(0,1)}$

We obtain a nontoric configuration for the surface $S^5_{(0,2)}$ with invariants

$$K^2 = \frac{9}{5} \qquad h^0(-K) = 2 \qquad \rho = 5 \qquad n = 5$$

(C) $(\mathcal{C}8) :\ 2 \times \frac{1}{5}(1,2) \longrightarrow A_2 + \frac{1}{5}(1,2) \qquad \mathbb{P}(1,3,5)$

The blow ups from $\mathbb{P}(1,3,5)$ gives the toric configuration for $S^{2,2}_{(0,2)}$ with invariants:

$$K^2 = \frac{24}{5} \qquad h^0(-K) = 5 \qquad \rho = 2 \qquad n = 2$$

Thus we end up with the minimal surfaces:

(I) $S^{2,1}_{(0,2)}$ and $S^{2,2}_{(0,2)}$ with $\rho = 2$ and $K^2 = \frac{24}{5}$

(II) $S^5_{(0,2)}$ with $\rho = 5$ and $K^2 = \frac{9}{5}$

Analysis for case (1,2)

Case 1 $(\mathcal{C}5) + (\mathcal{C}1) :\ \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \qquad S_{(1,1)}$

There are three cases of surfaces of type $(1,1)$ with $\rho = 3$, but from each of them we end up with a surface with a floating curve, thus none of these models are minimal.

Case 2 $(\mathcal{C}5) + (\mathcal{C}5) + (\mathcal{C}1) + (\mathcal{C}7) :\ \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow \text{smooth} \qquad \mathbb{P}^2$

The MMP in not directed as there is a $(\mathcal{C}1)$ contraction available before $(\mathcal{C}7)$.

Case 3 $(\mathcal{C}5) + (\mathcal{C}5) + (\mathcal{C}7) :\ \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow\ A_1 \qquad \mathbb{P}(1,1,2)$

We obtain the non toric configuration for the surface $S^{3,1}_{(1,2)}$ with invariants

$$K^2 = \frac{32}{15} \qquad h^0(-K) = 2 \qquad \rho = 4 \qquad n = 3$$

Case 4 (A) $(\mathcal{C}5)+(\mathcal{C}6)+(\mathcal{C}1)+(\mathcal{C}8):\ \frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2)\longrightarrow A_1+A_2 \qquad \mathbb{P}(1,2,3)$

The sequence gives two possible configurations with (respectively) a floating (-1)-curve and an available $(\mathcal{C}1)$ contraction.

(B) $(\mathcal{C}5)+(\mathcal{C}6)+(\mathcal{C}1):\ \frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2)\longrightarrow A_1+\frac{1}{5}(1,2) \qquad \mathbb{P}(1,2,5)$

This gives the non–toric configuration for $S^{3,2}_{(1,2)}$ with invariants:

$$K^2=\frac{32}{15} \qquad h^0(-K)=2 \qquad \rho=4 \qquad n=3$$

Case 5 $(\mathcal{C}5)+(\mathcal{C}6)+(\mathcal{C}1)+(\mathcal{C}9):\ \frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2)\longrightarrow \text{smooth} \qquad \mathbb{P}^2$

The MMP is not directed as there is a $(\mathcal{C}8)$ contraction available before $(\mathcal{C}9)$

Case 6 $(\mathcal{C}5)+(\mathcal{C}6)+(\mathcal{C}9):\ \frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2)\longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

Similarly as above, the MMP is not minimal because of an available $(\mathcal{C}1)$ contraction before $(\mathcal{C}9)$

Case 7 $(\mathcal{C}5)+(\mathcal{C}7):\ \frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2)\longrightarrow \frac{1}{5}(1,2) \qquad S_{(0,1)}$

We have two possibilities for minimal surfaces of type $(0,1)$:

(A) $S^3_{(0,1)}$: the sequence gives a $(\mathcal{C}5)$ contraction available before $(\mathcal{C}7)$

(B) $S^4_{(0,1)}$: similarly, we have a $(\mathcal{C}1)$ contraction available

Case 8 $(\mathcal{C}5)+(\mathcal{C}9):\ \frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2)\longrightarrow \frac{1}{3}(1,1) \qquad \mathbb{P}(1,1,3)$

We obtain the non toric configuration for $S^2_{(1,2)}$ with invariants:

$$K^2=\frac{62}{15} \qquad h^0(-K)=3 \qquad \rho=3 \qquad n=2$$

Case 9 $(\mathcal{C}5)+(\mathcal{C}10):\ \frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2)\longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

The resulting configuration is the same as case (8), so the MMP is not directed.

Case 10 $(\mathcal{C}5)+(\mathcal{C}12)+(\mathcal{C}3):\ \frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2)\longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

We get two non directed configurations with respectively $(\mathcal{C}5)$ and $(\mathcal{C}6)$ contractions available before $(\mathcal{C}12)$

Case 11 $(\mathcal{C}6)+(\mathcal{C}1):\ \frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2)\longrightarrow 2\times\frac{1}{5}(1,2) \qquad S_{(0,2)}$

We have two candidates of type $(0,2)$:

(A) $S^{2,1}_{(0,2)}$: get a configuration where a $(\mathcal{C}5)$ contraction is available

(B) $S^{2,2}_{(0,2)}$: the resulting configuration for $S^{3,3}_{1,2}$ is minimal with invariants

$$K^2 = \frac{32}{15} \qquad h^0(-K) = 2 \qquad \rho = 4 \qquad n = 3$$

Case 12 $(\mathcal{C}6) + (\mathcal{C}8) + (\mathcal{C}2) + (\mathcal{C}8)$: $\frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow A_1 + A_2 \qquad \mathbb{P}(1,2,3)$

The configuration contains a floating (-1)-curve, so not minimal.

Case 13 (A) $(\mathcal{C}6) + (\mathcal{C}8) + (\mathcal{C}2) + (\mathcal{C}9)$: $\frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow$ smooth $\qquad \mathbb{P}^2$

The configuration admits a $(\mathcal{C}5)$ contraction before $(\mathcal{C}9)$

(B) $(\mathcal{C}6) + (\mathcal{C}8) + (\mathcal{C}2)$: $\frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow$ smooth $\qquad \mathbb{P}(1,2,5)$

The sequence gives same configuration as case (A) above

Case 14 $(\mathcal{C}6) + (\mathcal{C}9)$: $\frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow \frac{1}{5}(1,2) \qquad S_{(0,1)}$ Similarly to case (7) we have two candidates:

(A) $S^{3}_{(0,1)}$: the configuration admits a $(\mathcal{C}5)$ contraction available

(B) $S^{4}_{(0,1)}$: a contraction of type $(\mathcal{C}8)$ is available before $(\mathcal{C}9)$

Case 15 $(\mathcal{C}6) + (\mathcal{C}13) + (\mathcal{C}4)$: $\frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow A_1 \qquad \mathbb{P}(1,1,2)$

The resulting MMP is not directed as a $(\mathcal{C}1)$ contraction is available before $(\mathcal{C}4)$

Case 16 $(\mathcal{C}8) + (\mathcal{C}2)$: $\frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \qquad S_{(1,1)}$

For all three cases of type $(1,1)$ we obtain a contraction of type $(\mathcal{C}5)$ available before $(\mathcal{C}8)$, so the MMP is not directed

Case 17 $(\mathcal{C}13) + (\mathcal{C}4)$: $\frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow \frac{1}{3}(1,1) \qquad \mathbb{P}(1,1,3)$

After the blow ups we have a $(\mathcal{C}5)$ contraction available, so the MMP is not directed.

Thus we end up with the minimal surfaces:

(I) $S^{2}_{(1,2)}$ with $\rho = 3$ and $K^2 = \frac{47}{15}$

(II) $S^{3,1}_{(1,2)}$, $S^{3,2}_{(1,2)}$ and $S^{3,3}_{(1,2)}$ with $\rho = 4$ and $K^2 = \frac{32}{15}$

Analysis for case (2,2)

Case 1 $(\mathcal{C}5) + (\mathcal{C}1): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \qquad S_{(2,1)}$

We have two candidates for surfaces of type $(2,1)$ with $\rho \leq 2$:

(A) $S^{1,1}_{(2,1)}$: get the configuration $S^{2,1}_{(2,2)}$ with invariants:

$$K^2 = \frac{22}{15} \qquad h^0(-K) = 1 \qquad \rho = 4 \qquad n = 2$$

(B) $S^{1,2}_{(2,1)}$: get the configuration $S^{2,2}_{(2,2)}$ with invariants:

$$K^2 = \frac{22}{15} \qquad h^0(-K) = 1 \qquad \rho = 4 \qquad n = 2$$

Case 2 $(\mathcal{C}5) + (\mathcal{C}6) + (\mathcal{C}7): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow A_1 + \frac{1}{5}(1,2) \qquad \mathbb{P}(1,2,5)$

We obtain two configurations which are isomorphic to $S^{2,1}_{(2,2)}$ and $S^{2,1}_{(2,2)}$.

Case 3 $(\mathcal{C}5) + (\mathcal{C}6) + (\mathcal{C}12): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow 2 \times A_1 + A_3 \qquad M$

Same configuration as case (2), so the MMP is not directed.

Case 4 $(\mathcal{C}5) + (\mathcal{C}7): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \qquad S^2_{(1,1)}$

We obtain a configuration with a $(\mathcal{C}6)$ contraction before $(\mathcal{C}7)$ available.

Case 5 $(\mathcal{C}6) + (\mathcal{C}1): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \qquad S_{(1,2)}$

There is no surface of type $(1,2)$ with $\rho \leq 2$.

Case 6 (A) $(\mathcal{C}6) + (\mathcal{C}6) + (\mathcal{C}9): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow A_1 + \frac{1}{5}(1,2) \qquad \mathbb{P}(1,2,5)$

We get the nontoric configuration where a $(\mathcal{C}5)$ contraction is available, thus the MMP is not directed.

(B) $(\mathcal{C}6) + (\mathcal{C}6): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow (\mathcal{F}0) + (\mathcal{F}4)$ The sequence gives the same configuration as the above case (A), so the MMP is not directed.

Case 7 $(\mathcal{C}6) + (\mathcal{C}7): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{5}(1,2) \qquad S^2_{(0,2)}$

The configuration will contain a $(\mathcal{C}6)$ contraction available before $(\mathcal{C}7)$

Case 8 $(\mathcal{C}6) + (\mathcal{C}8) + (\mathcal{C}10): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow A_1 + A_2 \qquad \mathbb{P}(1,2,3)$

We obtain two configurations that already appear in cases above.

Case 9 $(\mathcal{C}6) + (\mathcal{C}9): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \qquad S^2_{(1,1)}$

The resulting configuration has a $(\mathcal{C}6)$ contraction available before $(\mathcal{C}9)$.

Case 10 $(\mathcal{C}6) + (\mathcal{C}10): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow A_1 + \frac{1}{5}(1,2) \qquad \mathbb{P}(1,2,5)$

The sequence gives the toric configuration $S^1_{(2,2)}$ with invariants

$$K^2 = \frac{37}{15} \qquad h^0(-K) = 2 \qquad \rho = 3 \qquad n = 1$$

Case 11 $(\mathcal{C}6) + (\mathcal{C}12) + (\mathcal{C}13): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow A_1 + \frac{1}{5}(1,2) \qquad \mathbb{P}(1,2,5)$

We get a configuration where a contraction of type $(\mathcal{C}5)$ is available.

Case 12 $(\mathcal{C}6) + (\mathcal{C}13): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow A_1 + A_4 + \frac{1}{3}(1,1) \qquad \mathbb{P}(2,3,5)$

The configuration will admit a contraction of type $(\mathcal{C}9)$ before $(\mathcal{C}13)$ available.

Case 13 $(\mathcal{C}8) + (\mathcal{C}2): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \qquad S_{(2,1)}$

There are two candidates:

(A) $S^{1,1}_{(2,1)}$: get the configuration $S^{2,3}_{(2,2)}$ with invariants:

$$K^2 = \frac{22}{15} \qquad h^0(-K) = 1 \qquad \rho = 4 \qquad n = 2$$

(B) $S^{1,2}_{(2,1)}$: get the configuration $S^{2,4}_{(2,2)}$ with invariants:

$$K^2 = \frac{22}{15} \qquad h^0(-K) = 1 \qquad \rho = 4 \qquad n = 2$$

Case 14 $(\mathcal{C}10): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow \frac{1}{3}(1,1) + \frac{1}{5}(1,2) \qquad S_{(1,1)}$

There are four possibilities for surfaces of type $(1,1)$ and $\rho \leq 3$, but all of the resulting configurations will admit a contraction of type $(\mathcal{C}5)$ or $(\mathcal{C}6)$, so the MMPs are not directed.

Case 15 $(\mathcal{C}12) + (\mathcal{C}12): \ 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{5}(1,2) \longrightarrow 2 \times (\mathcal{F}0)$

The configuration will have a $(\mathcal{C}8)$ contraction available, so the MMP is not directed.

Thus we end up with the minimal surfaces:

(I) $S^1_{(2,2)}$ with $\rho = 3$ and $K^2 = \frac{37}{15}$

(II) $S^{2,i}_{(2,2)}$ $i = 1..4$ with $\rho = 4$ and $K^2 = \frac{22}{15}$

Analysis of case (3,2)

Case 1 $(\mathcal{C}10):\ 3\times\frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2)\longrightarrow 2\times\frac{1}{3}(1,1)+\frac{1}{5}(1,2)\qquad S_{(2,1)}$

From the two cases of surface of type $(2,1)$ with $\rho\leq 2$, we obtain the following configurations:

(A) $S^{1,1}_{(2,1)}$: toric surface $S^{0,1}_{(3,2)}$ with invariants

$$K^2=\frac{9}{5}\qquad h^0(-K)=1\qquad \rho=3\qquad n=0$$

(B) $S^{1,1}_{(2,1)}$: toric surface $S^{0,2}_{(3,2)}$ with invariants

$$K^2=\frac{9}{5}\qquad h^0(-K)=1\qquad \rho=3\qquad n=0$$

Case 2 (A) $(\mathcal{C}11)+(\mathcal{C}12):\ 3\times\frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2)\longrightarrow A_2+A_3+\frac{1}{5}(1,2)\qquad \mathbb{P}(3,4,5)$

The resulting configurations are the same as case (1)

(B) $(\mathcal{C}11):\ 3\times\frac{1}{3}(1,1)+2\times\frac{1}{5}(1,2)\longrightarrow(\mathcal{F}2)+(\mathcal{F}3)$ Again, we get the same configurations, so these sequences are not directed.

Thus we end up with the minimal surfaces:

(I) $S^{0,1}_{(3,2)}$ and $S^{0,2}_{(3,2)}$ with $\rho=3$ and $K^2=\frac{9}{5}$

Analysis of case (0,3)

Case 1 $(\mathcal{C}5)+(\mathcal{C}1):\ 3\times\frac{1}{5}(1,2)\longrightarrow 2\times\frac{1}{5}(1,2)\qquad S_{(0,2)}$

From the two cases of surface of type $(0,2)$ with $\rho\leq 2$, we obtain the following configurations:

(A) $S^{2,1}_{(0,2)}$: non toric surface $S^{3,1}_{(0,3)}$ with invariants

$$K^2=\frac{6}{5}\qquad h^0(-K)=1\qquad \rho=4\qquad n=3$$

(B) $S^{2,2}_{(0,2)}$: toric surface $S^{3,2}_{(0,3)}$ with invariants

$$K^2=\frac{6}{5}\qquad h^0(-K)=1\qquad \rho=4\qquad n=3$$

Case 2 $(\mathcal{C}5)+(\mathcal{C}5)+(\mathcal{C}1):\ 3\times\frac{1}{5}(1,2)\longrightarrow A_1+\frac{1}{5}(1,2)\qquad \mathbb{P}(1,2,5)$

The resulting configuration is not minimal as other type of contractions are available.

Case 3 $(\mathcal{C}5)+(\mathcal{C}5)+(\mathcal{C}9):\ 3\times\frac{1}{5}(1,2)\longrightarrow A_1\qquad \mathbb{P}(1,1,2)$

The configuration admits a $(\mathcal{C}1)$ contraction, so the surface is not minimal.

Case 4 $(\mathcal{C}5)+(\mathcal{C}8)+(\mathcal{C}2):\ 3\times\frac{1}{5}(1,2)\longrightarrow A_1+\frac{1}{5}(1,2)\qquad \mathbb{P}(1,2,5)$

The sequence is not directed. hence the surface is not minimal.

Case 5 $(\mathcal{C}5)+(\mathcal{C}9):\ 3\times\frac{1}{5}(1,2)\longrightarrow \frac{1}{5}(1,2)\qquad S^3_{(0,1)}$

The MMP is not directed as there is another $(\mathcal{C}5)$ contraction available before $(\mathcal{C}9)$

Case 6 $(\mathcal{C}5)+(\mathcal{C}13)+(\mathcal{C}4):\ 3\times\frac{1}{5}(1,2)\longrightarrow A_1\qquad \mathbb{P}(1,1,2)$

As above, a $(\mathcal{C}5)$ contraction is available.

Case 7 $(\mathcal{C}8)+(\mathcal{C}2):\ 3\times\frac{1}{5}(1,2)\longrightarrow 2\times\frac{1}{5}(1,2)\qquad S_{(0,2)}$

In both cases of type $(0,2)$ and $\rho=2$ there are contraction of type $(\mathcal{C}5)$ in the resulting configuration.

Case 8 $(\mathcal{C}8)+(\mathcal{C}8)+(\mathcal{C}2):\ 3\times\frac{1}{5}(1,2)\longrightarrow A_2+\frac{1}{5}(1,2)\qquad \mathbb{P}(1,3,5)$

From the sequence we get two configurations where (in both of them) a $(\mathcal{C}5)$ contraction is available, thus the MMP is not directed.

Case 9 $(\mathcal{C}14)+(\mathcal{C}4):\ 3\times\frac{1}{5}(1,2)\longrightarrow \frac{1}{5}(1,2)\qquad S^3_{(0,1)}$

The resulting configuration has a $(\mathcal{C}5)$ contraction available, so the surface is not minimal.

Thus we end up with the minimal surfaces:

(I) $S^{3,1}_{(0,3)}$ and $S^{3,2}_{(0,3)}$ with $\rho=4$ and $K^2=\frac{6}{5}$

Analysis of case (1,3)

Case 1 $(\mathcal{C}5)+(\mathcal{C}10):\ \frac{1}{3}(1,1)+3\times\frac{1}{5}(1,2)\longrightarrow A_1+\frac{1}{5}(1,2)\qquad \mathbb{P}(1,2,5)$

The sequence gives the non toric configuration $S^{1,1}_{(1,3)}$ with invariants

$$K^2=\frac{23}{15}\qquad h^0(-K)=1\qquad \rho=3\qquad n=1$$

Case 2 $(\mathcal{C}5) + (\mathcal{C}13): \ \frac{1}{3}(1,1) + 3 \times \frac{1}{5}(1,2) \longrightarrow A_1 + A_4 + \frac{1}{5}(1,2) \qquad \mathbb{P}(2,3,5)$

Obtain same configuration as case (1)

Case 3 $(\mathcal{C}8) + (\mathcal{C}10): \ \frac{1}{3}(1,1) + 3 \times \frac{1}{5}(1,2) \longrightarrow A_1 + \frac{1}{5}(1,2) \qquad \mathbb{P}(1,3,5)$

Get two distinct configurations:

(A) $S^{1,2}_{(1,3)}$ with non toric with invariants

$$K^2 = \frac{23}{15} \qquad h^0(-K) = 1 \qquad \rho = 3 \qquad n = 1$$

(B) A nontoric configuration that is isomorphic to $S^{1,1}_{(1,3)}$

From the case $(\mathcal{C}8): \ \frac{1}{3}(1,1) + 3 \times \frac{1}{5}(1,2) \longrightarrow (\mathcal{F}2) + (\mathcal{F}3)$ get same configurations.

Case 4 $(\mathcal{C}8) + (\mathcal{C}12): \ \frac{1}{3}(1,1) + 3 \times \frac{1}{5}(1,2) \longrightarrow A_2 + A_3 + \frac{1}{5}(1,2) \qquad \mathbb{P}(3,4,5)$

The sequence gives three distinct configurations:

(A) $S^{1,3}_{(1,3)}$: non toric with invariants

$$K^2 = \frac{23}{15} \qquad h^0(-K) = 1 \qquad \rho = 3 \qquad n = 1$$

(B) A nontoric configuration isomorphic to $S^{1,2}_{(1,3)}$

(C) A nontoric configuration isomorphic to $S^{(1,3)}_{(1,3)}$

Case 5 $(\mathcal{C}10): \ \frac{1}{3}(1,1) + 3 \times \frac{1}{5}(1,2) \longrightarrow 2 \times \frac{1}{5}(1,2) \qquad S_{(0,2)}$

From both cases of type $(0,2)$ we have configurations with a $(\mathcal{C}8)$ contraction available.

Case 6 $(\mathcal{C}12): \ \frac{1}{3}(1,1) + 3 \times \frac{1}{5}(1,2) \longrightarrow (\mathcal{F}0) + (\mathcal{F}3)$

Get same configurations as case (3).

Thus we end up with the minimal surfaces:

(I) $S^{1,1}_{(0,3)}$, $S^{1,2}_{(0,3)}$ and $S^{1,3}_{(0,3)}$ with $\rho = 3$ and $K^2 = \frac{23}{15}$

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